POINTWISE GREEN FUNCTION BOUNDS AND STABILITY OF COMBUSTION WAVES

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Abstract. Generalizing similar results for viscous shock and relaxation waves, we establish sharp pointwise Green function bounds and linearized and nonlinear stability for traveling wave solutions of an abstract viscous combustion model including both Majda's model and the full reacting compressible Navier–Stokes equations with artificial viscosity with general multi-species reaction and reaction-dependent equation of state, under the necessary conditions of strong spectral stability, i.e., stable point spectrum of the linearized operator about the wave, transversality of the profile as a connection in the traveling-wave ODE, and hyperbolic stability of the associated Chapman–Jouguet (square-wave) approximation. Notably, our results apply to combustion waves of any type: weak or strong, detonations or deflagrations, reducing the study of stability to verification of a readily numerically checkable Evans function condition. Together with spectral results of Lyng and Zumbrun, this gives immediately stability of small-amplitude strong detonations in the small heat-release (i.e., fluid-dynamical) limit, simplifying and greatly extending previous results obtained by energy methods by Liu–Ying and Tesei–Tan for Majda's model and the reactive Navier–Stokes equations, respectively.

1. Introduction

In this paper, we extend the viscous shock stability theory of [ZH, MaZ2, MaZ3, MaZ4, HZ] to traveling waves of combustion models, including the simplified combustion model of Majda, and an artificial viscosity version of the reacting Naver-Stokes equations. Specifically, we (i) derive sharp pointwise Green function bounds, yielding a sharp $L^1 \cap L^p \to L^p$ linearized stability criterion in terms of an Evans function condition, and (ii) assuming the Evans stability condition, establish nonlinear stability for waves of arbitrary type: weak or strong detonation, weak of strong deflagration.

The results described in this paper, Theorems 1.2 and 1.5 below, represent in particular the first stability results of any kind for large-amplitude combustion waves and for weak detonations of Majda's model.

This reduces the question of linear and nonlinear stability to verification of the Evans condition, a criterion that is readily checked numerically [Br1, Br2, Br3, BrZ, BDG].

1.1. Combustion models.

We show that viscous shock and combustion waves, like their hyperbolic counterparts, can be studied within a common framework. Indeed, viscous shocks, viscous detonations, and relaxation shocks may all be considered as traveling waves of the

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special class of hyperbolic–parabolic balance laws, or reaction–diffusion–convection equations,

$$U_t + \mathcal{F}(U) = 0, \qquad \mathcal{F}(U) = F(U)_x - (B(U)U_x)_x - G(U), \tag{1.1}$$

having the damping property

$$\Re\, \sigma(dG) \leq 0, \tag{1.2}$$

where (here and elsewhere) $\sigma(M)$ denotes spectrum of a matrix or linear operator M. For viscous shocks, $G \equiv 0$, while for relaxation shocks, dG has constant rank, its kernel corresponding to a local equilibrium manifold.

By contrast, combustion equations have the composite structure

$$G(U) = \phi(U)\tilde{G}(U),$$

where ϕ is a scalar "ignition function" that turns the reaction on or off– specifically, it is zero on some subset of the state space and positive elsewhere– and $\tilde{G}$ is a relaxation type term, $d\tilde{G}$ has constant rank and $\sigma(d\tilde{G}) \leq 0$: that is, an interpolation between the viscous and relaxation case. Thus, traveling combustion waves exhibit features of *both* viscous and relaxation shocks, in various different regimes, and our analysis must take this into account.

Specifically, we study a subclass of (1.1), comprising systems of the form,

$$\begin{cases} u_t + f(u,z)_x = bu_{xx} + qk\phi(u)z, \\ \qquad\qquad z_t = dz_{xx} - k\phi(u)z, \end{cases} \tag{1.3}$$

where $u \in \mathbb{R}^n$ and $z \in \mathbb{R}^r$, and ϕ is a "bump"-type ignition function. The physical constant q is the heat release parameter. Here, $q > 0$ corresponds to an exothermic reaction.

When $n = 1$ and $r = 1$, (1.3) is Majda's single-reaction combustion model. Then, u is a lumped variable combining various aspects of specific volume, particle velocity, and temperature, while $z \in [0,1]$ is the mass fraction of reactant. The positive constant k represents the rate of the reaction. In Majda's model, the diffusion coefficients b and d are also assumed to be positive constants. In the following, we scale the variables so that $b \equiv 1$.

The vectorial version of (1.3), with $u \in \mathbb{R}^n$, $z \in \mathbb{R}^r$, and b and d positive definite matrices, is sufficient to encompass the artificial viscosity version of the full reactive compressible Navier–Stokes equations written in Lagrangian coordinates, with multi-species reaction and reaction-dependent equation of state, where $u = (\tau, v, E)$, with τ, v, and E denoting specific volume, velocity, and energy density, $\phi = \phi(T)$, $z_1, \ldots, z_r$ denoting mass fractions of reactant species, and k matrix-valued with eigenvalues of strictly negative real part [Z1, LyZ2].

Throughout the paper, we shall carry out in parallel the analysis of the scalar and the (artificial viscosity) system case, exposing the main ideas in the simpler setting of Majda's model, then indicating by a series of brief remarks the extension to the general case.

Physical (as opposed to artifical) diffusion terms are of form $(b(u,z)u_x)_x$ and $(d(u,z)z_x)_x$ with b, d matrix-valued and b semi-definite [LyZ1, LyZ2]. The diffusion coefficient b is commonly assumed to depend on u alone; however, like the equation of state, it properly depends on the make-up of the gas, hence on the mass fraction z of the reactant. See comments in section 1.3, about the extension of the results of this paper to such systems.

1.2. **Statement of the results.** Consider a general traveling-wave solution $U(x,t)=\bar{U}(x-st)$ of (1.1), and the associated linearized equation about $\bar{U}$ in moving coordinates $(x-st,t)$:

$$U_t + LU = 0, \qquad LU := (\mathcal{F}'(\bar{U}) - s\partial_x)U. \tag{1.4}$$

Proposition 3.2 below, adapted from [LyZ2], discusses the question of the existence of traveling waves for the scalar version of (1.3); regarding the system version, see Remark 3.6.2.

Definition 1.1. *Let X and Y be two Banach spaces. A traveling wave $\bar{U}$ solution of* (1.1) *is said to be $X \to Y$ linearly orbitally stable if, for any solution $\tilde{U}$ of* (1.4) *with initial data in X, there exists a phase shift δ, such that $\tilde{U}(\cdot,t)$ approaches $\delta(t)\bar{U}'(\cdot)$, in Y and as $t\to\infty$.*

Our first main theorem is the following linearized stability criterion, precisely analogous to that of the viscous shock case, in terms of the Evans function $D(\cdot)$ associated with the linearized operator L about the wave, an analytic function defined on frequencies $\lambda : \Re\lambda \geq 0$, whose zeroes correspond to eigenvalues of L (see Section 5.3 for further details).

Theorem 1.2. *A traveling combustion wave of* (1.3) *is $L^1\cap L^p \to L^p$ linearly orbitally stable for $p>1$ if and only if*

$$D(\cdot) \text{ has precisely one zero in } \{\Re\lambda \geq 0\} \text{ (necessarily at } \lambda = 0). \tag{1.5}$$

Theorem 1.2 is obtained as a result of detailed pointwise bounds on the Green function of the linearized equations about the wave, given in Proposition 7.1 (resp. Remark 7.2.4 in the system case); see Section 7.3.

Note that the spatial derivative $\bar{U}'(x)$ of the traveling-wave profile $\bar{U}$ is always a zero eigenfunction of L, a consequence of translation invariance of the original evolution equation. Thus, at a formal level, condition (1.5) corresponds to the statement that perturbations in all directions other than translation decay with time to the order of linear approximation, or linearized orbital stability. (At the formal level only, due to the absence of spectral gap between $\sigma(L)$ and the origin $\lambda=0$; see [ZH] for further discussion.)

More precisely, it was shown in [LyZ1, LyZ2], that, similarly as in the viscous shock case [ZS],

$$D(\lambda) = \gamma\Delta(\lambda) + o(|\lambda|) \tag{1.6}$$

for $|\lambda|$ sufficiently small, where γ is a constant, and Δ is a homogeneous degree one Lopatinski determinant.

The constant γ measures the angle between the unstable subspace at $-\infty$ and the stable subspace at $+\infty$ for the traveling wave ODE (in this paper, equations (3.1)-(3.3)), that is, the *transversality* of the traveling wave as a solution of the traveling wave ODE.

The condition $\Delta(\lambda)\neq 0$ for $\Re\lambda\geq 0$ and $\lambda\neq 0$ is equivalent to linear stability of the corresponding inviscid shock (square wave approximation) as a solution of the hyperbolic Chapman-Jouguet equations (the Chapman-Jouguet limit is the instantaneous reaction limit, or $k=+\infty$, of the inviscid $(b,d=0)$ limit of (1.3)).

Thus, similarly as in the viscous shock or relaxation case, condition (1.5) is equivalent to,

$$
\left\{
\begin{array}{r}
\sigma(L) \subset \{\Re\lambda \leq 0\} \cup \{0\}, \\
\gamma \neq 0, \\
\Delta(\lambda) \neq 0 \text{ for } \Re\lambda \geq 0 \text{ and } \lambda \neq 0,
\end{array}
\right.
\tag{1.7}
$$

that is, strong spectral stability (first condition in (1.7)), plus transversality, plus Lopatinksi stability of the associated square-wave (Chapman–Jouguet) approximation.

Note that (1.6) holds in the much more general multidimensional case as well [Z1, JLW].

Remark 1.3. It is shown in [JLW] that, under "standard" assumptions of a reaction-indepenent, ideal gas equation of state, strong detonations are always Chapman–Jouguet stable. Together with (1.6), this has the interesting consequence that transition from viscous stability to instability as parameters are varied must occur either by breakdown of transversality in the traveling-wave connection, or else by crossing of the imaginary axis of one or more nonzero complex conjugate eigenvalue pairs, i.e., a Poincaré–Hopf type bifurcation. This agrees with physically observed "galloping" or "pulsating" instabilities; see [LyZ2, TZ1, TZ2] for further discussion.

Definition 1.4. *Let X and Y be two Banach spaces. A traveling wave solution $\bar{U}$ of* (1.1) *is said to be $X \to Y$ nonlinearly orbitally stable if, for any solution $\tilde{U}$ of* (1.1) *with initial data sufficiently close in X to $\bar{U}$, there exists a phase shift δ, such that $\tilde{U}(\cdot, t)$ approaches $\bar{U}(\cdot - \delta(t))$, in Y and as $t \to \infty$. If, also, the phase $\delta(t)$ converges to a limiting value $\delta(+\infty)$, the profile is said to be nonlinearly phase-asymptotically orbitally stable.*

Using the information given by Theorem 1.2, we further obtain our second main theorem, asserting that strong spectral stability implies nonlinear stability. This is a corollary of the more detailed, pointwise version given in Theorem 1.5, in which we let,

$$\hat{L}^\infty := \{f \in \mathcal{S}'(\mathbb{R}), \quad (1 + |\cdot|)^{3/2} f(\cdot) \in L^\infty\}.$$

In particular, $\hat{L}^\infty \hookrightarrow L^1 \cap L^p$, for all $1 \leq p \leq +\infty$.

Theorem 1.5. *Under condition* (1.5)*, a traveling combustion wave $\bar{U}(x - st)$ of* (1.3) *is $\hat{L}^\infty \to L^p$ nonlinearly phase-asymptotically orbitally stable, for $p > 1$. More precisely, given $\bar{U}$ a traveling-wave solution of* (1.3)*, given $1 \leq p \leq \infty$, there exist $E_0 > 0$, $C > 0$, $\delta(\cdot) \in C^1$, and $\delta(+\infty) \in \mathbb{R}$, such that the unique solution $\tilde{U}$ of* (1.1) *issuing from the initial datum $\bar{U} + U_0$, where,*

$$(1 + |x|)^{3/2} |U_0(x)| \leq E_0,$$

satisfies the asymptotic estimates,

$$
\begin{aligned}
|\tilde{U}(x, t) - \bar{U}(x - \delta(t))|_{L^p} &\leq C E_0 (1 + t)^{-\frac{1}{2}(1 - \frac{1}{p})}, \\
|\dot{\delta}(t)| &\leq C E_0 (1 + t)^{-1}, \\
|\delta(t) - \delta(+\infty)| &\leq C E_0 (1 + t)^{-1/2}.
\end{aligned}
\tag{1.8}
$$

1.3. **Comments.** We indicate in this section how the above theorems relate to previous mathematical results on combustion waves.

Strong detonations are combustion waves for which the underlying gas dynamical shock is of Lax type (see section 3.1).

Nonlinear stability of small-amplitude strong detonations in the small-q limit was established by Li, Liu and Tan [LLT] using spectral analysis together with Sattinger's method of weighted norms [Sa] and by Liu and Ying [LYi] using energy estimates, for Majda's model. Nonlinear stability of strong detonation in the small-q limit for the related Majda-Rosales model (where z_t is replaced by z_x in the reaction equation), with explicit rates of convergence, was established by Li [Li1]. Nonlinear stability of small-amplitude strong detonations in the small-q limit for full reactive Navier–Stokes (with Heaviside-type ignition function, and reaction-independent equation of state) was established by Tan and Tesei in [TT] using detailed energy estimates.

Roquejoffre and Vila [RV] studied spectral stability of arbitrary amplitude strong detonations in the small-k (ZND) limit, for Majda's model (in the case $d = 0$). Together with the weighted norm argument of [LLT], this is sufficient to yield nonlinear stability with time-exponential rate, for exponentially decaying initial data.

Most recently, Lyng and Zumbrun [LyZ1] have shown by an elementary perturbation argument using an abstract Evans function framework that spectral stability of strong detonations in the small-q (i.e., fluid-dynamical) limit amounts to spectral stability of the underlying gas-dynamical shock.

For strong detonations, in the Majda model case, the spectral results of [LLT] yield (1.5) and thus full linearized and nonlinear stability. We note that, even though the weighted norm method suffices (as pointed out in [LLT]) to yield a nonlinear stability result for strong detonations with exponentially decaying initial perturbations, our result applies to much more general (in particular, algebraically decaying) data and yields additional pointwise detail on solution structure.

As in the shock wave case, our approach yields ultimately the *reduction* of stability analysis to a spectral problem. Thus, the following result is a consequence of Theorem 1.5, together with the spectral analysis of Lyng and Zumbrun [LyZ1]. This greatly extends and simplifies the strongest prior result of [TT], illustrating the power of the method.

Corollary 1.6. *Strong detonation waves of* (1.3) *are* $\hat{L}^\infty \to L^p$ *nonlinearly orbitally stable,* $p > 1$*, in small-q limit if and only if the limiting gas-dynamical shock is stable; in particular, for Majda's model, they are always stable.*

Remark 1.7. It would be interesting to extend the spectral analysis of [RV] to the case $d \neq 0$, which would then imply nonlinear stability of arbitrary strength strong detonations for Majda's model in the small-k (ZND) limit. More interesting still would be to extend this to the system case. We conjecture that the proper system analog, similar to the small-q result of [LyZ1], is that stability in the ZND limit is equivalent to gas-dynamical stability of the component Neumann shock (see discussion, [LyZ1, GS1, GS2]) together with hyperbolic (i.e., Lopatinski) stability of the associated ZND detonation. This should be provable by a combination of the singular perturbation methods of [PZ, FS] and "multi-pulse" calculations carried out for multiple traveling-pulse solutions in models of nerve-impulse and optical transmission. This would recover the [RV] result of stability for the scalar, Majda model, for which the gas-dynamical shock, since scalar, is automatically stable (see, e.g., [Sa]) and

the ZND detonation may readily be calculated to be stable. For systems, however, ZND detonations are often unstable, so that stability in the ZND limit should not be expected.

Weak detonations are combustion waves for which the underlying gas dynamical shock is undercompressive (see section 3.1).

Nonlinear stability of weak detonations was obtained by Szepessy [S] for small-amplitude waves with intermediate k, and by Liu and Yu [LY] for arbitrary amplitude waves in the large-k limit. Both these papers deal with the Majda-Rosales system.

As far as we know, Theorem 1.5 is the first analytical result on nonlinear stability of weak detonations for the Majda model (more generally, the vectorial version including reactive Navier–Stokes equations with artificial viscosity), and also for deflagrations of any type.[1]

Deflagrations, weak and strong, are other types of "undercompressive" combustion waves.

It would be very interesting to determine (presumably by numerical computations) the existence (here assumed) and stability or instability (that is, verification of condition (1.5)) of weak or strong deflagrations.

Finally, note that abstract one-dimensional stability results on deflagrations are likely not to be so physically important, since multi-d transverse instabilities appear to play such a prominent role in their behavior [B].

As noted in [M], one-dimensional deflagration waves feature a pressure and a velocity which are locally nearly constant. Then, a "constant density" approximation [MS] shows that the interaction between the chemical reaction and fluid dynamics may be neglected. That is, roughly speaking, the complicated equations modeling reacting gas decouple into a part describing the fluid flow and a part describing the chemical reaction. As a result, deflagration waves are often modeled as systems of reaction–diffusion equations.

The fact that detonations are usually approximated by reaction–convection equations (ZND), and deflagrations by reaction–diffusion equations, reflects the general belief that these are dominating effects in the two different contexts. Our analysis here via reaction–convection–diffusion puts both on the same footing, allowing treatment in a unified theoretical/numerical framework, *investigation/validation* of these beliefs, determination of their realms of validity.

1.5. **Notes on the proof.** An important aspect of the Lax shock analysis is that differentiated source leads to faster temporal decay [Z3, MaZ3]. Where $\phi \geq c_0 > 0$, note as in relaxation case that source in nonequibrium mode gives faster-decaying response, at differentiated rate, and so can be treated as in relaxation case. Where $\phi = 0$, undifferentiated source does not appear, and so can be treated as in usual shock case. What makes this technically feasible is that, near traveling waves, the two regions are spatially separated, corresponding to $x \leq -M$ and $x \geq M$, $M > 0$, respectively. The intermediate regime $c_0 > \phi \geq 0$ is localized within the internal layer, corresponding undifferentiated source appears with exponentially decaying multiplier $e^{-\theta|x|}$, $\theta > 0$. But, sources of the latter order appear already in the undercompressive shock case, and can be treated by the methods of [HZ] with no change.

[1]In particular, we note that the weighted norm technique of [LLT] is inherently restricted to strong detonations for the scalar, Majda model; see the discussion of [TZ1].

So, our analysis is by interpolation between the viscous undercompressive shock analysis of [HZ, MaZ3] and the relaxation shock analysis of [MaZ1]. The new aspects of the argument not present in the undercompressive shock case are isolated to bound (8.8)(ii), Remark 8.2, and the new auxiliary Lemma 8.6. In particular, no new convolution estimates were necessary, only a series of observations having to do with the fact that undifferentiated sources appear in a direction for which the Green function decays more rapidly, at differentiated rate.

Remarks 1.8. 1. As a relaxation system, (1.3) is somewhat degenerate, violating the usual assumption of genuine coupling between equilibrium and relaxation variables u and z associated with time-asymptotic smoothing of solutions (see, e.g., [MaZ1, Z2] and references therein). Indeed, asymptotic decoupling of the z-equation plays an important role in the analysis; see Remarks 4.1 and 4.4.2. Diffusion terms b, d not present in standard relaxation systems enforce smoothing directly.

2. The case $d = 0$ that is often considered for Majda's model requires slightly different handling. Absence of z-diffusion leads to "hyperbolic" delta-function components in reactive modes reminiscent of those encountered in [MaZ1, MaZ3] in the case of relaxation or degenerate viscosity, but with the difference that incoming modes on side $x \geq 0$ are not time-exponentially damped. This can be accomodated in the analysis by the introduction of an exponentially weighted norm in the spirit of [Sa] in the z-component only, for $x \geq 0$, using the property of exponential decoupling as $x \to 0$ of reactive and fluid modes. This is a mathematical issue only; for physical models, d is strictly parabolic: $\Re\sigma(d) > 0$.

1.6. **Extension to the Navier-Stokes equations with physical viscosity.** The full reactive Navier–Stokes equations with real, or physical viscosity may be treated by essentially the same techniques, using the more complicated arguments (and more detailed Green fn. bounds) developed in [MaZ3, MaZ4, Z2, R, HRZ] for the treatment of viscous shocks with real viscosity. However, these arguments so far are limited to the strong detonation case. (Likewise, for technical reasons, the viscous shock theory is so far limited for physical viscosity to the Lax and overcompressive case.) We leave this to a future work.

1.7. **Plan of the paper.** In Section 2, we describe Majda's combustion model and its vectorial generalization, in Section 3 the various types of traveling wave connections that may occur, and in Section 4 the linearized eigenvalue equations about these traveling waves. In Sections 5 and 6, we construct the Evans function and resolvent kernel of the Linearized operator about the wave following the abstract framework of [ZH, MaZ3], specializing in the low-frequency regime to the special structure of (1.3) using the limiting constant-coefficient calculations of Section 4. In Section 7, we convert the resulting pointwise resolvent kernel bounds to pointwise Green function bounds by stationary phase type estimates on the Inverse Laplace transform formula, in the process establishing Theorem 1.2 equating linearized and spectral (Evans) stability. Finally, in Section 8, we carry out a nonlinear stability analysis, establishing Theorem 1.5.

2. Preliminaries

2.1. **Majda's model.** We begin with the scalar version of system (1.3). We assume as in [LyZ2] that f, $\phi \in C^2$,

$$f_u(u,z) > 0, \quad f_{uu}(u,z) > 0, \tag{2.1}$$

and that ϕ is a "bump"-type ignition function that is identically zero for $u \le u_i$ or $u \ge u^i$ and strictly positive for $u_i < u < u^i$.

It is sometimes useful to rewrite (1.3) in the conservative form

$$(u+qz)_t + f(u,z)_x = u_{xx} + qdz_{xx}, \tag{2.2}$$

$$z_t = dz_{xx} - k\phi(u)z. \tag{2.3}$$

Remarks 2.1. 1. Note that the flux f, modeling equation of state, depends on z, modeling the chemical constituation of the gas, with the linear dependence loosely following the averaged equation of state derived in [CHT] for the full Euler equations. This is important for realistic modeling of the full equations of reacting flow; see [CHT, LyZ2] for further discussion. For the Majda model, new qualitative phenomena emerge for $f_z \not\le 0$ at the levels of both existence and behavior of detonation profiles [LyZ2].

2. Following [M], ϕ is usually taken to be a "step"-type function, vanishing for $u \le u_i$ and positive for $u > u_i$. As discussed in [LyZ2], our alternative choice of a bump-type function is motivated by the physical parametrization of temperature with respect to velocity u in the traveling-wave phase portrait of the ZND model. This choice admits all the phemomena of the step-ignition case, restricting to $u < u^i$. In addition, it allows for existence of weak deflagration profiles (defined in Section 3), as the step-type ignition function in general does not; see [LyZ2] or Remark 3.1.

2.2. **Reacting Navier–Stokes equations.** The single-species reacting Navier–Stokes equations with artificial viscosity, written in Lagrangian coordinates, take the form

$$\begin{aligned} \tau_t - v_x &= 0, \\ v_t + p_x &= b_1 v_{xx} \\ E_t + (pv)_x &= b_2 E_{xx} + q_3 k\phi(T)z, \\ z_t &= dz_{xx} - k\phi(T)z, \end{aligned} \tag{2.4}$$

where τ, v, $E = e + qz + v^2/2$, z denote specific volume (ρ^{-1}, where ρ is density), velocity, total energy density, and mass fraction of reactant, $T = T(\tau, e)$ temperature, and $p = p(\tau, e, z)$ pressure, $k, q_3, b_j, d > 0$ constant, or

$$q = \begin{pmatrix} 0 \\ 0 \\ q_3 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & b_2 \end{pmatrix}$$

in (1.3). The ignition function ϕ is assumed to vanish identically for T below some critical ignition temperature T_i, and to be strictly positive for T above T_i.

A common choice of equation of state is the reaction-independent gamma-law

$$p = \Gamma e/\tau, \qquad T = c^{-1}e, \tag{2.5}$$

where c is the specific heat constant and $\Gamma = \gamma - 1$ is the Gruneisen constant. In the thermodynamical rarified gas approximation, $\gamma > 1$ is the average over constituent particles of $\gamma = (n+2)/n$, where n is the number of internal degrees of freedom of an individual particle [Ba].

A more accurate assumption, following [CHT], is to view the gas as a composite of unburned and burned phases with *different* equations of state $T_j(\tau_j, e_j)$, $p_j(\tau_j, e_j)$, $j = 1$ corresponding to the unburned and $j = 2$ to the burned state, with

$$\begin{aligned} &\tau_1 = \tau/z, \quad \tau_2 = \tau/(1-z) \qquad (\text{i.e., } \rho_1 = z\rho, \quad \rho_2 = (1-z)\rho), \\ &T_1 = T_2 = T, \quad e = e_1 + e_2, \quad p = p_1 + p_2. \end{aligned} \tag{2.6}$$

If both phases obey (different) gamma-law equations of state, this leads [CHT] to a gamma-law-type equation of state (2.5) with reaction-dependent coefficients

$$c(z) = zc_1 + (1-z)c_2, \qquad \Gamma(z) = \frac{zc_1\Gamma_1 + (1-z)c_2\Gamma_2}{zc_1 + (1-z)c_2}. \tag{2.7}$$

This is the "typical" equation of state we have in mind.

3. Traveling Waves

We consider traveling-wave solutions, i.e., solutions of the form

$$u(x,t) = \bar{u}(x - st), \quad z(x,t) = \bar{z}(x - st), \quad s > 0,$$

of (1.3) that connect an unburned state $(u_+, z_+) = (u_+, 1)$ to a completely burned state $(u_-, z_-) = (u_-, 0)$. These are combustion waves that move from left to right leaving completely burned gas in their wake.

Thus, dropping bars for notational convenience, we find that the traveling-wave Ansatz leads, after an integration, from (1.3) to the system of ordinary differential equations:

$$u' = f(u,z) - f(u_-, z_-) - qdy - sqz - s(u - u_-), \tag{3.1}$$

$$z' = y, \tag{3.2}$$

$$y' = d^{-1}\big(-sy + k\phi(u)z\big), \tag{3.3}$$

where we have used $y := z'$ to write the system in first order and $'$ denotes differentiation with respect to the variable $\xi := x - st$. We assume that the end states are such that

$$u_- \in [u_i, u^i], \quad u_+ \notin [u_i, u^i], \tag{3.4}$$

so that

$$\phi(u_-) > 0, \quad \phi(u_+) = 0, \quad \phi'(u_+) = 0. \tag{3.5}$$

Equation (3.4) has the physical interpretation that the unburned end state is below ignition temperature so that the there is no chemical reaction on the unburnt side.

3.1. **Rankine–Hugoniot conditions.**

A necessary condition for the existence of a connection is that the end states at $\pm\infty$ be rest points of the traveling-wave equation. This leads to the Rankine-Hugoniot condition

$$f(u_+, z_+) - f(u_-, z_-) = sq + s(u_+ - u_-), \tag{RH}$$

together with the requirements that

$$y_\pm = z'_\pm = 0 \tag{3.6}$$

and (justifying assumptions (3.5))

$$\phi(u_\pm)z_\pm = 0. \tag{3.7}$$

Restricting now to Majda's model, write $\hat{\alpha}_\pm := f_u(u_\pm, z_\pm)$ and $\beta_\pm := f_z(u_\pm, z_\pm)$. Then, the traveling-wave profile is said to be a *strong detonation* if

$$\hat{\alpha}_- > s > \hat{\alpha}_+. \tag{3.8}$$

It is said to be a *weak detonation* if

$$s > \hat{\alpha}_-,\, \hat{\alpha}_+. \tag{3.9}$$

Similarly, it is said to be a *weak deflagration* if

$$\hat{\alpha}_-, \hat{\alpha}_+ > s. \tag{3.10}$$

It is said to be a *strong deflagration* if

$$\hat{\alpha}_+ > s > \hat{\alpha}_-. \tag{3.11}$$

Degenerate profiles for which the inequalities are nonstrict are called *Chapman–Jouguet* detonations or deflagrations and lie on the boundary between weak and strong branches.

Remark 3.1. For f independent of z, we find by (2.1) that detonations correspond to case $u_- > u_i \geq u_+$, deflagrations to case $u_- \leq u^i < u_+$. In particular, deflagrations cannot occur for a step-type ignition function, for which $u^i = +\infty$.

A routine modification of Propositions 2.1 and 2.2, [LyZ2], accounting for z-dependence of f, yields the following description of solutions of (RH).

Proposition 3.2. *For fixed u_+, suppose that $f(u_+, 1) < f(u_+ + q, 0)$. Then, there exist $s^*(u_+) < s_*(u_+)$ such that (i) for $s > s_*$ there exist two states $u_- > u_+$ for which* (RH) *(but not necessarily* (3.5)*) is satisfied (weak and strong detonation), for $s = s_*$ there exists one (Chapman–Jouguet detonation), and for $s < s_*$, there exist none no solutions $u_- > u_+$. (ii) For $s < s^*$, there exist two states $u_- < u_+$ for which* (RH) *is satisfied (weak and strong deflagration), for $s = s^*$, there exists one (Chapman–Jouguet deflagration), and for $s > s^*$, there exist none. If $f(u_+, 1) \geq f(u_+ + q, 0)$, on the other hand, then for each s there exists at most one solution $u_- > u^+$ and one solution $u_- < u^+$, (strong detonation and strong deflagration, respectively).*[2]

Remark 3.3. The case $f(u_+, 1) < f(u_+ + q, 0)$ is essentially identical to that of the reaction-independent case discussed in [LyZ2], for which $f(u_+) < f(u_+ + q)$ by (2.1). More generally, $f_z \leq 0$ is sufficient for $f(u_+, 1) < f(u_+ + q, 0)$. For further discussion of the (RH) problem, see [LyZ2]: in particular the Chapman–Jouguet diagrams of Figure 1.

3.2. **The connection problem.** Linearizing (3.1)–(3.3) around the state (u_-, z_-, y_-), we find the constant-coefficient system of ordinary differential equations

$$\begin{pmatrix} u \\ z \\ y \end{pmatrix}' = \begin{pmatrix} \hat{\alpha}_- - s & b_- - sq & -qd \\ 0 & 0 & 1 \\ 0 & kd^{-1}\phi(u_-) & -sd^{-1} \end{pmatrix} \begin{pmatrix} u \\ z \\ y \end{pmatrix}. \tag{3.12}$$

For strong detonations, the coefficient matrix in (3.12) is easily seen to have two positive eigenvalues and one negative eigenvalue. Thus, there is a two-dimensional unstable manifold

[2]Typically, both; in particular, if f grows superlinearly in $|u|$, then both solutions exist for each s.

at $(u_-,0,y_-)$. Similarly, we note that the linearization of (3.1)–(3.3) about (u_+,z_+,y_+) is

$$\begin{pmatrix} u \\ z \\ y \end{pmatrix}' = \begin{pmatrix} \hat\alpha_+ - s & b_+ - sq & -qd \\ 0 & 0 & 1 \\ 0 & 0 & -sd^{-1} \end{pmatrix} \begin{pmatrix} u \\ z \\ y \end{pmatrix}. \tag{3.13}$$

By the block-triangular structure, it is easy to see that there are two negative eigenvalues and one zero eigenvalue. It is easy to see that the center manifold is a line of equilibria, so plays no orbit may approach the rest point (u_+,z_+,y_+) along the center manifold. Thus, for connections, the important structure is the two-dimensional stable manifold at $(u_+,1,y_+)$. Counting dimensions, we see that a connection corresponds to the intersection of two two-dimensional manifolds in $\mathbb{R}^3$. In particular, it generically persists as a unique, transverse intersection, under variations in parameters such as $u_\pm$, s consistent with (RH). See [LyZ2] for a discussion of this situation in the case $d=0$.

Similarly, for weak detonations, there is a one-dimensional unstable manifold at $(u_-,0,y_-)$ and a two-dimensional stable manifold at $(u_+,0,y_+)$. Thus, connections are typically *codimension one* in the set of (RH) compatible parameters, in contrast to the strong detonation case. See [LyZ2] in the case $d=0$. This situation is analogous to that of a Lax-type shock in the nonreactive case. For weak deflagrations, there is a a two-dimensional unstable manifold at $(u_-,0,y_-)$ and a one-dimensional stable manifold at $(u_+,0,y_+)$. Thus, connections are again generically *codimension one* in the set of (RH) compatible parameters. See [LyZ2] in the case $d=0$. For strong deflagrations, there is a a one-dimensional unstable manifold at $(u_-,0,y_-)$ and a one-dimensional stable manifold at $(u_+,0,y_+)$, and connections are *codimension two.*

In every case, we have by the discussion above:

Lemma 3.4. *Traveling-wave profiles $(\bar u,\bar z)$ corresponding to weak or strong detonations or deflagrations satisfy*

$$|(d/dx)^k\Big((\bar u,\bar z)-(u,z)_\pm\Big)|\le Ce^{-\theta|x|}, \qquad x\gtrless 0, \quad 0\le k\le 3. \tag{3.14}$$

Proof. Standard ODE estimates for stable and unstable manifolds. □

Remark 3.5. Weak detonations and deflagrations are analogous to undercompressive shocks in the nonreactive case, with strong deflagrations undercompressive of degree two; see [Z2] for a discussion of shock classification. In the case $d=0$, it can be demonstrated that weak detonation connections do occur in some cases, but deflagration connections (weak or strong) of the type we have described do not [LyZ2].[3] It is an interesting question whether or not they occur for $d\neq 0$, or, more generally, for the full, reactive Navier–Stokes equations [LyZ1, LyZ2].

3.3. **The system case.** Under the further assumption of (asymptotic) *dissipativity*,

$$\Re\sigma(\hat\alpha i\xi-\xi^2 b)_\pm \le \frac{-\theta|\xi|^2}{1+|\xi|^2}, \qquad \hat\alpha_\pm := (\partial f/\partial u)(U_\pm), \tag{3.15}$$

$\theta>0$, for all $\xi\in\mathbb{R}$ (standard for systems [Z2]), it is readily verified using the above-mentioned block-triangular decomposition of limiting systems into fluid and reactive blocks

[3]More precisely, weak deflagration profiles exist only in the degenerate case $u_+=u^i$, with $u(x)$ converging to u_- as $x\to+\infty$ at subalgebraic rate; strong deflagration profiles do not exist in any case [LyZ2].

together with fluid dynamical results of Majda and Pego [MP], that the main results of this section carry over to the system case, substituting for definitions (3.8)–(3.11) the system versions (for right-going waves, $s > 0$)

$$\hat\alpha_n^- > s > \hat\alpha_n^+, \quad s > \hat\alpha_j^\pm,\ j \neq n, \qquad \text{(strong detonation)} \tag{3.16}$$

$$s > \hat\alpha_n^-,\ \hat\alpha_n^+, \quad s > \hat\alpha_j^\pm,\ j \neq n, \qquad \text{(weak detonation)} \tag{3.17}$$

$$\hat\alpha_n^-, \hat\alpha_n^+ > s, \quad s > \hat\alpha_j^\pm,\ j \neq n, \qquad \text{(weak deflagration)} \tag{3.18}$$

and

$$\hat\alpha_n^+ > s > \hat\alpha_n^-. \quad s > \hat\alpha_j^\pm,\ j \neq n, \qquad \text{(strong deflagration)} \tag{3.19}$$

where $\hat\alpha_1^\pm < \cdots < \hat\alpha_n^\pm$ denote the eigenvalues of $\hat\alpha_\pm := f_u(u_\pm, z_\pm)$.

In particular, connections if they exist satisfy (RH), and if they are transverse are generically of codimension zero, one, one, two, respectively, in the set of (RH)-compatible parameters. Further, the connecting profile satisfies (3.14), converging exponentially to its endstates $U_\pm$ as $x \to \pm\infty$. See [Z1], Appendix A for further details. Condition (3.15) holds trivially for identity viscosity $b = I$, and holds also for the physical (semidefinite) viscosity of the reacting Navier–Stokes equations [MaZ4, Z2], the two main cases we have in mind. For simplicity of notation, we assume also that $\sigma(d)$ is semisimple, so that d has a full set of eigenvectors.

Likewise, there is a simple analogy to the Chapman–Jouget analysis of Section 3.1, and, for the typical mixed gamma-law equation of state (2.7) of Proposition 3.2. For, rearranging (RH) in the case of the reacting Navier–Stokes equations (2.4), we obtain [LyZ1] from the first equation that $(v_+ - v_-) = -s(\tau_+ - \tau_-)$, from the second that $(\tau, p)_\pm$ lie on the Rayleigh line

$$p_+ - p_- = -s^2(\tau_+ - \tau_-), \tag{3.20}$$

and from the third the shifted Hugoniot curve

$$(e_+ - e_-) + q = (-1/2)(p_+ + p_-)(\tau_+ - \tau_-). \tag{3.21}$$

Thus, fixing $(\tau, v, E)_+$, viewing (3.21) as determining a "burned" pressure law

$$p_- = P_-(\tau_-), \tag{3.22}$$

and assuming that e_+ can be recovered from τ_-, p_- through inversion of $p_- = p(\tau_-, e_-, 0)$, we find that the allowable states $(\tau, v, E)_-$ are determined as the intersection in the $\tau - p$ plane of line (3.20) with curve (3.22), similarly as in the scalar (Majda's model) case. Moreover, there are two distinct solution structures, according as

$$p_+ < P_-(\tau_+) \tag{3.23}$$

(standard: for P_- convex, pairs of weak/strong detonations, deflagrations as in the scalar case) or the reverse (nonstandard: for P_- convex, single strong detonation, deflagration).

For the typical equation of state (2.5)–(2.7), it is readily calculated that

$$P_-(\tau) = \frac{\big(\tau_+/\Gamma_1 - (1/2)(\tau - \tau_+)\big)p_+ + q}{\tau/\Gamma_2 - (1/2)(\tau - \tau_+)}, \tag{3.24}$$

whence (3.23) reduces to

$$p_+(1-\Gamma_2/\Gamma_1) < q\Gamma_2/\tau_+. \tag{3.25}$$

From (3.25), we see that $\Gamma_1 \le \Gamma_2$ (in particular, including the reaction-independent case) implies a standard (RH) solution structure. Roughly speaking, this corresponds to a reaction in which complicated compounds break up into simpler components, so that n decreases and $\Gamma = 2/n$ increases, the reverse situation to a reaction in which simple components combine into more complicated molecules. We conjecture, by analogy with the scalar case, that for (2.7), condition (3.25) equivalent to $\Gamma_z \le 0$ implies further a standard connection structure, at least in the ZND limit (for which there is a close connection to Majda's model [LyZ1, LyZ2, GS1]). However, even for $\Gamma_1 > \Gamma_2$, the standard (RH) solution structure is recovered for q sufficiently large.

Remarks 3.6. 1. Dissipativity, (3.15), implies in particular hyperbolicity of the first-order convection terms, i.e., that α has real, semisimple eigenvalues. We use this freely below.

2. Existence of detonation connections for the full, reacting Navier–Stokes equations has been studied by Gasser and Szmolyan [GS1, GS2] using geometric singular perturbation techniques in the ZND (b, $d \to 0$) limit, and by Gardner [G] using Conley index methods.

4. The eigenvalue equation

Suppose $\big(\bar u(x-st), \bar z(x-st)\big)$ is a traveling-wave profile of (1.3) as described above. We now begin to investigate the stability of such an object. The linearized equations about $(\bar u, \bar z)$ in moving coordinates $\tilde x = x - st$, are, dropping tildes,

$$u_t - q(k\phi'(\bar u)u\bar z + k\phi(\bar u)z) + (\alpha u)_x + (\beta z)_x = u_{xx}, \tag{4.1}$$

$$z_t - sz_x = -k\phi'(\bar u)u\bar z - k\phi(\bar u)z + dz_{xx}, \tag{4.2}$$

where $\alpha := f_u(\bar u, \bar z) - s$, $\beta := f_z(\bar u, \bar z)$, and u, z now denote perturbations. The eigenvalue equations corresponding to this linear system are thus

$$u'' = \big(\lambda u - q(k\phi'(\bar u)u\bar z + k\phi(\bar u)z) + (\alpha u)' + (\beta z)'\big), \tag{4.3}$$

$$z'' = d^{-1}\big(\lambda z - sz' + k\phi'(\bar u)u\bar z + k\phi(\bar u)z\big), \tag{4.4}$$

Alternatively, upon substituting $dz'' - \lambda z + sz' = k\phi'(\bar u)u\bar z + k\phi(\bar u)z$ from (4.4) into (4.3), we can rewrite (4.3) as

$$u'' = \big(\lambda(u+qz) - sqz' - qdz'' + (\alpha u)' + (\beta z)'\big). \tag{4.5}$$

Compare this with the remark at the end of Section 2. We write (4.3)–(4.4) as a first-order system. To do so, we define $W := (u, z, u', z')^{\mathrm{tr}}$, so that (4.3)–(4.4) becomes

$$W' = \mathbb{A}(x,\lambda)W \tag{4.6}$$

where the coefficient matrix is

$$\mathbb{A}(x,\lambda) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda + \alpha' - qk\phi'(\bar u)\bar z & \beta' - qk\phi(\bar u) & \alpha & \beta \\ d^{-1}k\phi'(\bar u)\bar z & d^{-1}\lambda + d^{-1}k\phi(\bar u) & 0 & -sd^{-1} \end{pmatrix}. \tag{4.7}$$

System (4.6) has a limiting constant-coefficient structure, i.e., the coefficient matrix has limits as $x \to \pm\infty$. That is,

$$\mathbb{A}(x,\lambda) \to \mathbb{A}_\pm(\lambda) \quad \text{as} \quad x \to \pm\infty,$$

and the limiting matrices are given by

$$\mathbb{A}_\pm(\lambda) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda & -kq\phi(u_\pm) & \alpha_\pm & \beta_\pm \\ 0 & d^{-1}(\lambda + k\phi(u_\pm)) & 0 & sd^{-1} \end{pmatrix}. \tag{4.8}$$

Remark 4.1. Here, we coordinatize as $W = (u, z, u', z')^{\mathrm{tr}}$ following the general scheme of [ZH, MaZ3] as this is what we shall need below to establish the pointwise bounds. However, it is sometimes helpful to see the fluid/reaction structure in the system. To see this at the level of the eigenvalue ODEs, we write

$$\hat{W} := (u, u', z, z')^{\mathrm{tr}}, \tag{4.9}$$

separating the fluid (u, u') and reaction (z, z') quantities. In this labeling scheme, the eigenvalue ODE (4.6) becomes

$$\hat{W}' = \hat{\mathbb{A}}(x,\lambda)\hat{W} \tag{4.10}$$

where the matrix $\hat{\mathbb{A}}$ can easily be obtained from (4.7) by appropriately swapping entries, and

$$\hat{\mathbb{A}}_\pm(\lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \lambda & \alpha_\pm & -kq\phi(u_\pm) & \beta_\pm \\ 0 & 0 & 0 & 1 \\ 0 & 0 & d^{-1}(\lambda + k\phi(u_\pm)) & -sd^{-1} \end{pmatrix}. \tag{4.11}$$

In particular, the upper block-triangular structure of $\hat{\mathbb{A}}_\pm$ will be useful in the constant-coefficient analysis below.

4.1. **Constant–coefficient analysis.** We now examine the constant-coefficient limiting systems $W' = \mathbb{A}_\pm(\lambda)W$ (or equivalently $\hat{W}' = \hat{\mathbb{A}}_\pm(\lambda)\hat{W}$) in some detail. From the upper block-triangular structure in (4.11), it is quite straightforward to compute eigenvalues; they are simply the eigenvalues of the diagonal blocks. From the upper left-hand "fluid" block, we obtain the fluid eigenvalues

$$\mu_f^\pm = \frac{\alpha_\pm \pm \sqrt{\alpha_\pm^2 + 4\lambda}}{2}, \tag{4.12}$$

while the lower right-hand "reaction" block contributes eigenvalues of form

$$\mu_r^+ = \frac{-sd^{-1} \pm \sqrt{s^2 d^{-2} + 4d^{-1}\lambda}}{2} \quad \text{from} \quad \hat{\mathbb{A}}_+, \tag{4.13}$$

and

$$\mu_r^- = \frac{-sd^{-1} \pm \sqrt{s^2 d^{-2} + 4(d^{-1}\lambda + d^{-1}k\phi(u_-))}}{2} \quad \text{from} \quad \hat{\mathbb{A}}_-. \tag{4.14}$$

The corresponding eigenvectors also have structure inherited from the block-triangular nature of the limiting matrices $\hat{\mathbb{A}}_\pm$. In particular, as long as the fluid and reaction eigenvalues

remain distinct — as our calculations below show they are for small λ, the corresponding eigenvectors take the form

$$\hat{v}_f^\pm = \begin{pmatrix} 1 \\ \mu_f^\pm \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \hat{v}_r^\pm = \begin{pmatrix} M_\pm^{-1} \begin{pmatrix} 1 \\ \mu_r^\pm \end{pmatrix} \\ 1 \\ \mu_r^\pm \end{pmatrix}, \tag{4.15}$$

where

$$M_+(\lambda) := \left[\begin{pmatrix} 0 & 1 \\ \lambda & \alpha_+ \end{pmatrix} - \mu_r^+ I \right]^{-1} \begin{pmatrix} 0 & 0 \\ 0 & \beta_+ \end{pmatrix} \tag{4.16}$$

and

$$M_-(\lambda) := \left[\begin{pmatrix} 0 & 1 \\ \lambda & \alpha_- \end{pmatrix} - \mu_r^- I \right]^{-1} \begin{pmatrix} 0 & 0 \\ -qk\phi(u_-) & \beta_- \end{pmatrix}. \tag{4.17}$$

We also record here Taylor series expansions of those eigenvalues of the limiting systems which become zero at $\lambda = 0$, the so-called slow modes. These are

$$\mu_r^+(\lambda) = \frac{1}{s}\lambda - \frac{2d}{s^3}\lambda^2 + \cdots \tag{4.18}$$

and

$$\mu_f^\pm(\lambda) = \pm\frac{1}{\alpha_\pm}\lambda \mp \frac{1}{\alpha_\pm^3}\lambda^2 + \cdots, \tag{4.19}$$

with associated (right) eigenvectors, now written in (u, z, u', z') coordinates,

$$v_r^+(\lambda) = \begin{pmatrix} R_r^+ \\ 0 \end{pmatrix} + \cdots \tag{4.20}$$

and

$$v_f^\pm(\lambda) = \begin{pmatrix} R_f^\pm \\ 0 \end{pmatrix} + \cdots, \tag{4.21}$$

where limiting fluid modes

$$R_f^\pm = \begin{pmatrix} * \\ 0 \end{pmatrix} \tag{4.22}$$

have vanishing z-component.

We shall also have need of the adjoint eigenvalue equation

$$\begin{aligned} \tilde{u}'' &= \lambda^* \tilde{u} - qk\phi'(\bar{u})\bar{z}\tilde{u} + k\phi'(\bar{u})\bar{z}\tilde{z} - \alpha\tilde{u}', \\ \tilde{z}'' &= d^{-1}\big(\lambda^* - qk\phi(\bar{u})\tilde{u} + k\phi(\bar{u})\tilde{z} - \beta\tilde{u}' + s\tilde{z}'\big) \end{aligned} \tag{4.23}$$

associated with (4.3)–(4.4), where λ^* denotes complex conjugate. Writing as a first-order system $\tilde{W}' = \tilde{\mathbb{A}}(x, \lambda^*)\tilde{W}$, $\tilde{W} = (\tilde{u}, \tilde{z}, \tilde{u}', \tilde{z}')^{\mathrm{tr}}$, and studying the eigenvalues $\tilde{\mu}_j^\pm$ and eigenvectors $\tilde{v}_j^\pm$ of the limiting, coefficient-matrix $\tilde{\mathbb{A}}_\pm(\lambda^*)$, we have by duality that $\tilde{\mu}_j^\pm = (\mu_j^\pm)^*$, while a brief calculation yields Taylor expansion

$$\tilde{v}_f^\pm(\lambda) = \begin{pmatrix} L_f^\pm \\ 0 \end{pmatrix} + \cdots, \qquad \tilde{v}_r^\pm(\lambda) = \begin{pmatrix} L_r^\pm \\ 0 \end{pmatrix} + \cdots \tag{4.24}$$

at $\lambda = 0$, where limiting left fluid modes $L_f^- = c(1, q)^{\mathrm{tr}}$ on the minus infinity (reactive) side satisfy

$$L_f^- \perp (-q, 1)^{\mathrm{tr}}. \tag{4.25}$$

(See Remark 4.4.1 below for further discussion, and extension to the system case.)

Definition 4.2. The *domain of consistent splitting* for an ODE $W' = \mathbb{A}(x, \lambda)W$ with asymptotically constant coefficients is the set of $\lambda \in \mathbb{C}$ such that

(i) the limiting matrices $\mathbb{A}_\pm(\lambda)$ are hyperbolic, i.e., they have no center subspace, and
(ii) the dimensions of the stable (unstable) subspaces $S^+(\lambda)$ and $S^-(\lambda)$ ($U^+(\lambda)$ and $U^-(\lambda)$) are the same.

Lemma 4.3. *The set $\{\lambda \in \mathbb{C} : \Re\lambda > 0\}$ is a subset of the domain of consistent splitting.*

Proof. Immediate by inspection of formulas (4.12)–(4.14). □

In fact, more can be said. The limiting matrices $\hat{\mathbb{A}}_\pm$ fail to be hyperbolic if

$$\det(\hat{\mathbb{A}}_\pm(\lambda) - i\xi) = 0$$

for some $\xi \in \mathbb{R}$. But, again using the upper block-triangular structure of $\hat{\mathbb{A}}_\pm$, it is clear that this determinant vanishes if and only if the determinant of one of the diagonal blocks vanishes. This leads to the following four *dispersion curves* in the complex λ-plane,

$$\begin{aligned}
\lambda_f^\pm(\xi) &= -\xi^2 - i\xi\alpha_\pm,\\
\lambda_r^+(\xi) &= d(-\xi^2 + isd^{-1}),\\
\lambda_r^-(\xi) &= d(-\xi^2 + isd^{-1}) - \phi(u_-).
\end{aligned}$$

These curves, parabolae opening into the left complex half plane, can be used to describe the boundary of the domain of consistent splitting. In particular, we note that Lemma 4.3 gives that the open right half complex plane is contained in the domain of consistent splitting, and, varying λ from right to left, we find that λ cannot leave the domain unless we cross one of these four dispersion curves. Thus, the component of the domain of consistent splitting which contains $+\infty$ contains a set of form

$$\Omega_\eta := \{\lambda : \Re\lambda > \max\{-\eta_1|\Im\lambda|, \eta_2|\Im\lambda|^2\}\}, \qquad \eta_j > 0.$$

See (6.10) and Lemma 6.2 below.

4.2. **The system case.** The above calculations extend in straightforward fashion to the system case, substituting block matrix computations for the scalar computations above. In particular, the block-triangular structure of (4.11) is maintained, reducing the computation of constant-coefficient modes to a computation on the upper lefthand diagonal fluid block that is exactly the viscous shock computation done in [ZH, MaZ3], and a computation on the lower righthand diagonal reaction block that on the plus infinity side is a particularly simple (scalar convection) version of the same viscous shock computation and on the minus infinity side consists of fast modes that need not be resolved.

Specifically, the Taylor expansions of slow modes at $\lambda = 0$ become

$$\mu_{r,i}^+(\lambda) = \frac{1}{s}\lambda - \frac{2d_j^+}{s^3}\lambda^2 + \cdots, \tag{4.26}$$

$i = 1, \dots, m$, $z \in \mathbb{R}^m$,

$$\mu^{\pm}_{f,j}(\lambda) = \pm \frac{1}{\alpha^{\pm}_j}\lambda \mp \frac{b^{\pm}_j}{(\alpha^{\pm}_j)^3}\lambda^2 + \cdots, \tag{4.27}$$

$j = 1, \dots, n$, $u \in \mathbb{R}^n$,

$$v^{+}_{r,i}(\lambda) = \begin{pmatrix} R^{+}_{r,i} \\ 0 \end{pmatrix} + \cdots \qquad R^{\pm}_{r,i} = \begin{pmatrix} * \\ r_{r,i} \end{pmatrix}, \tag{4.28}$$

and

$$v^{\pm}_{f}(\lambda) = \begin{pmatrix} R^{\pm}_{f,j} \\ 0 \end{pmatrix} + \cdots, \qquad R^{\pm}_{f,j} = \begin{pmatrix} r_{f,j} \\ 0 \end{pmatrix}, \tag{4.29}$$

$$\tilde{v}^{\pm}_{f}(\lambda) = \begin{pmatrix} L^{\pm}_{f,j} \\ 0 \end{pmatrix} + \cdots, \qquad L^{+}_{f,j} = c\begin{pmatrix} l^{+}_{f,j} \\ 0 \end{pmatrix}, \qquad L^{-}_{f,j} = c\begin{pmatrix} l^{-}_{f,j} \\ q^{\mathrm{tr}} l^{-}_{f,j} \end{pmatrix}, \tag{4.30}$$

where $l^{+}_{r,i}$ and $r_{r,i}$ are left and right eigenvectors of d (now matrix-valued) and $\alpha^{\pm}_{f,j}$, $l^{\pm}_{f,j}$, $r^{\pm}_{f,j}$ are the eigenvalues and left and right eigenvectors of $\partial f/\partial u(U_{\pm})$, $b^{\pm}_j = (l_{f,j} b r_{f,j})_{\pm}$ (*strictly positive*, by dissipativity assumption (3.15)), and $d^{+}_i = (l_{r,i} d r_{r,i})_{+}$. Note that we again have vanishing of the z-component of asymptotic fluid modes $R^{\pm}_{f,j}$, as well as the key orthogonality relation

$$L^{-}_{f,j} \perp \begin{pmatrix} -q \\ I_r \end{pmatrix}. \tag{4.31}$$

Remarks 4.4. 1. The structural relations (4.22), (4.25) and their system analogs (4.29)–(4.31) for the asymptotic modes, play an important role in our analysis; see Remarks 6.9.1, 8.7 and the proofs of Proposition 7.1, Lemma 8.6, and Proposition 8.1. Indeed, this is essentially the only structure that we use, other than the existence of Taylor expansions of slow modes at $\lambda = 0$. These may be verified easily by substituting into the limiting, constant-coefficient eigenvalue systems

$$\begin{aligned} u'' &= \big(\lambda u - qk\phi_{\pm} z + \alpha_{\pm} u' + \beta_{\pm} z'\big), \\ z'' &= d^{-1}\big(\lambda z - sz' + k\phi_{\pm} z\big) \end{aligned} \tag{4.32}$$

and their dual, adjoint systems the Ansatze $U = e^{\mu x} R$, $\tilde{U} = e^{\mu^* x} L$, respectively, to obtain characteristic equations

$$\begin{pmatrix} \mu^2 - \alpha_{\pm}\mu - \lambda I_n & qk\phi_{\pm} - \beta_{\pm}\mu \\ 0 & d\mu^2 - \lambda I_r + s\mu - k\phi_{\pm}) \end{pmatrix} R = 0$$

and

$$L^* \begin{pmatrix} \mu^2 + \alpha_{\pm}\mu - \lambda I_n & qk\phi_{\pm} + \beta_{\pm}\mu \\ 0 & d\mu^2 - \lambda I_r + s\mu - k\phi_{\pm}) \end{pmatrix} = 0,$$

respectively, which, setting $\lambda = \mu = 0$ to obtain slow mode behavior at $\lambda = 0$, reduce to

$$\begin{pmatrix} 0 & qk\phi_{\pm} \\ 0 & -k\phi_{\pm} \end{pmatrix} R = 0, \qquad L^* \begin{pmatrix} 0 & qk\phi_{\pm} \\ 0 & -k\phi_{\pm} \end{pmatrix} = 0,$$

from which (4.29)–(4.31) are evident.

2. The containment $\Omega_\eta \subset \Lambda$ likewise carries over to the system case, by dissipativity assumption (3.15) and Lemma 6.2 below.

5. The Evans function

We now construct the Evans function following the abstract framework of [MaZ3, Z2]. For historical origins of the Evans function, see [AGJ, PW, GZ] and references therein.

5.1. **The Conjugation Lemma.** We first recall a central result connecting variable- and constant-coefficient ODE. Consider a general family of first-order ODE

$$W' - \mathbb{A}(x,\lambda)W = 0 \tag{5.1}$$

of the form (4.6), indexed by a spectral parameter $\lambda \in \Omega \subset \mathbb{C}$, where $W \in \mathbb{C}^N$, $x \in \mathbb{R}$ and "$'$" denotes d/dx, assuming (cf. Lemma 3.4)

(h0) Coefficient $\mathbb{A}(\cdot,\lambda)$, considered as a function from Ω into $C^0(x)$ is analytic in λ. Moreover, $\mathbb{A}(\cdot,\lambda)$ approaches exponentially to limits $\mathbb{A}_\pm$ as $x \to \pm\infty$, with uniform exponential decay estimates

$$|(\partial/\partial x)^k(\mathbb{A} - \mathbb{A}_\pm)| \le C_1 e^{-\theta|x|/C_2}, \quad \text{for } x \gtrless 0,\ 0 \le k \le K, \tag{5.2}$$

C_j, $\theta > 0$, on compact subsets of Ω.

The following asymptotic ODE result generalizes the Gap Lemma of [GZ]; for a proof, see, e.g., [MZ1, MaZ3, Z2].

Proposition 5.1 (The Conjugation Lemma [MZ1])**.** *Given (h0), there exist locally to any given $\lambda_0 \in \Omega$ invertible linear transformations $P_+(x,\lambda) = I + \Theta_+(x,\lambda)$ and $P_-(x,\lambda) = I + \Theta_-(x,\lambda)$ defined on $x \ge 0$ and $x \le 0$, respectively, $\Phi_\pm$ analytic in λ as functions from Ω to $C^0[0,\pm\infty)$, such that:*

(i) For any fixed $0 < \bar\theta < \theta$ and $0 \le k \le K+1$, $j \ge 0$,

$$|(\partial/\partial\lambda)^j(\partial/\partial x)^k\Theta_\pm| \le C(j)C_1C_2e^{-\theta|x|/C_2} \quad \text{for } x \gtrless 0. \tag{5.3}$$

(ii) The change of coordinates $W =: P_\pm Z$ reduces (5.1) to

$$Z' - \mathbb{A}_\pm Z = 0 \quad \text{for } x \gtrless 0. \tag{5.4}$$

5.2. **Normal modes.** Using Proposition 5.1, we next construct normal modes for (5.1). Recall the domain of consistent splitting defined in Definition 4.2, Section 4.1.

Lemma 5.2. *On any simply connected subset of the domain of consistent splitting Λ, there exist analytic bases $\{V_1,\dots,V_k\}^\pm$ and $\{V_{k+1},\dots,V_N\}^\pm$ for the subspaces $S_\pm$ and $U_\pm$ defined in Definition 4.2.*

Proof. By spectral separation of $U_\pm$, $S_\pm$, the associated (group) eigenprojections are analytic. The existence of analytic bases then follows by a standard result of Kato; see [Kat], pp. 99–102. ∎

By Lemma 5.1, on the domain of consistent splitting, the subspaces

$$\mathcal{S}^+ = \text{span}\{W_1^+,\dots,W_k^+\} := \text{span}\{P_+V_1^+,\dots,P_+V_k^+\} \tag{5.5}$$

and

$$\mathcal{U}^- := \text{span}\{W_{k+1}^-,\dots,W_N^-\} := \text{span}\{P_-V_{k+1}^-,\dots,P_-V_N^-\} \tag{5.6}$$

uniquely determine the stable manifold as $x \to +\infty$ and the unstable manifold as $x \to -\infty$ of (5.1), defined as the manifolds of solutions decaying as $x \to \pm\infty$, respectively, independent

of the choice of $P_\pm$. More generally, $W_j^\pm := (PV_j)_\pm$, $j = 1, \dots, N$ are called *normal modes* for (5.1).

In the context of Majda's model (more generally, the system analog HERE), $V_j^\pm$ are comprised of the vectors described in Section 4.1 (resp. 4.2), of which the *slow modes*, defined as those approaching the center subspace of $\mathbb{A}_\pm$ as $\lambda \to 0$, are $v_f^\pm$, $v_r^\pm$ (resp. $v_{f,j}^\pm$, $v_{r,i}^\pm$). As *fast* growing and decaying modes, defined as those approaching the stable and unstable subspace of $\mathbb{A}_\pm$, hence spectrally separated both from each other and from slow modes, may always be chosen analytically in a neighborhood of $\lambda = 0$ by the same argument used in Lemma 5.2, we obtain by our asymptotic description of slow modes the following important extension.

Lemma 5.3. *For Majda's model (more generally, the vectorial version including reactive Navier–Stokes equations with artificial viscosity), normal modes extend to $\Lambda \cup B(0,r)$ for $r > 0$ sufficiently small, with low-frequency (slow mode) asymptotics as described in Section 4.1 (resp. 4.2).*

Proof. The explicit Taylor expansions of Section 4.1 (resp. 4.2) yield analytic extensions on $B(0,r)$ of slow modes $V_j^\pm$, spanning invariant subspaces of $\mathbb{A}_\pm$. As fast modes always have such analytic extensions, we may combine them to obtain analytic bases $V_j^\pm$ for invariant subspaces $S^\pm$, $U^\pm$ of $\mathbb{A}_\pm$ extending those of Lemma 5.2. These in turn determine $\mathbb{A}_\pm$-invariant projections onto those subspaces, which must therefore be the unique analytic extension of the corresponding projections on $\Lambda \cap B(0,r)$, and thus an analytic extension onto $\Lambda \cup B(0,r)$. The result then follows again by the result of Kato as in the proof of Lemma 5.2. ☐

Remark 5.4. The extension of normal modes through the essential spectrum boundary to a neighborhood of $\lambda = 0$ is crucial for all that follows; see [GZ, ZH] for further discussion.

5.3. **Construction of the Evans function.**

Definition 5.5. On any simply connected subset of the domain of consistent splitting, let $V_1^+, \dots, V_k^+$ and $V_{k+1}^-, \dots, V_N^-$ be analytic bases for S_+ and U_-, as described in Lemma 5.2. Then, the *Evans function* for (5.1) associated with this choice of limiting bases is defined as

$$
\begin{aligned}
D(\lambda) &:= \det\Big(W_1^+, \dots, W_k^+, W_{k+1}^-, \dots, W_N^-\Big)_{|x=0,\lambda} \\
&= \det\Big(P_+V_1^+, \dots, P_+V_k^+, P_-V_{k+1}^-, \dots, P_-V_N^-\Big)_{|x=0,\lambda},
\end{aligned}
\tag{5.7}
$$

where $P_\pm$ are the transformations described in Lemma 5.1.

Remark 5.6. Note that D is independent of the choice of $P_\pm$; for, by uniqueness of stable/unstable manifolds, the exterior products (minors) $P_+V_1^+ \wedge \cdots \wedge P_+V_k^+$ and $P_-V_{k+1}^- \wedge \cdots \wedge P_-V_N^-$ are uniquely determined by their behavior as $x \to +\infty, -\infty$, respectively.

Proposition 5.7 (MaZ3, Z)**.** *Both the Evans function and the stable/unstable subspaces $\mathcal{S}^+$ and $\mathcal{U}^-$ are analytic on the entire simply connected subset of the domain of consistent splitting on which they are defined. Moreover, for λ within this region, equation* (5.1) *admits a nontrivial solution $W \in L^2(x)$ if and only if $D(\lambda) = 0$.*

Remark 5.8. In the case that (5.1) describes an eigenvalue equation associated with an ordinary differential operator L, $\lambda \in \mathbb{C}^1$, Proposition 5.7 implies that eigenvalues of L agree

in location with zeroes of D. In [GJ1, GJ2], Gardner and Jones have shown that they agree also in multiplicity; see also Lemma 6.1, [ZH], or Proposition 6.15 of [MaZ3].

By Lemma 5.3, we have immediately that D extends to a neighborhood of the origin.

Lemma 5.9. *For Majda's model (more generally, the vectorial version including reactive Navier–Stokes equations with artificial viscosity), D extends analytically to $\Lambda \cup B(0,r)$ for $r > 0$ sufficiently small.*

6. The resolvent kernel

Next, we estimate the resolvent kernel $G_\lambda(x,y) := (L-\lambda)^{-1}\delta_y(x)$ associated with the linearized operator L about the wave, again following the abstract framework developed in [ZH, MaZ3], Rewriting (1.3) in vectorial form

$$U_t + F(U)_x + G(U) = BU_{xx}, \tag{6.1}$$

$$U := (u,z), \qquad F(U) = \begin{pmatrix} f(u,z) - su \\ -sz \end{pmatrix}, \qquad G(U) = \begin{pmatrix} \phi(u)qkz \\ -\phi(u)kz \end{pmatrix}, \qquad B = \begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix}, \tag{6.2}$$

in coordinates moving with a given traveling-wave profile $\bar U(x) = (\bar u, \bar z)(x)$ (stationary, in the moving coordinate frame) and linearizing about $\bar U$, we obtain the linearized equations

$$U_t = LU := BU_{xx} - (AU)_x + CU, \tag{6.3}$$

where

$$A := dF(U) = \begin{pmatrix} \alpha & 0 \\ 0 & -s \end{pmatrix}, \qquad B = \begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix}, \qquad C := dG(U) = \begin{pmatrix} qk\phi'(\bar u)\bar z & qk\phi(\bar u) \\ -k\phi'(\bar u)\bar z & -k\phi(\bar u) \end{pmatrix}.$$

The results of [ZH, MaZ3] for general strictly parabolic systems of the form (6.3), $U \in \mathbb{R}^n$, state that the resolvent kernel is a meromorphic function on the domain of consistent splitting defined in Section 4.1, where it is determined by

$$(L-\lambda)G_\lambda = \delta_y(x) \tag{6.4}$$

and the property of decay as x, $y \to \pm\infty$. Moreover, they give an explicit description of G_λ in terms of the normal modes of the eigenvalue equation constructed in Section 5.2, from which we may extract sharp pointwise bounds. We cite here the relevant theory, referring the reader to [ZH, MaZ3] for proof.

6.1. Duality relation.

Consider solutions U of the eigenvalue equation $(L-\lambda)U = 0$ and solutions $\tilde U$ of its adjoint $(L^* - \bar\lambda)\tilde U$, where

$$L^*\tilde U := B^{\mathrm{tr}}\tilde U_{xx} + A^{\mathrm{tr}}\tilde U_x + C^{\mathrm{tr}}\tilde U \tag{6.5}$$

denotes the L^2 adjoint of L and $\bar\lambda$ the complex conjugate of λ. Introducing the phase-variables $W := (U, U')$ and $\tilde W := (\tilde U, \tilde U')$, write the eigenvalue equation of L and its adjoint as first-order ODE of form (5.1) in W and $\tilde W$. Then, we have the following key relation.

Lemma 6.1 ([ZH, MaZ3]). *$W = (U, U')$ satisfies* (5.1) *if and only if*

$$\tilde{W}^* \mathcal{S} W \equiv \text{constant} \tag{6.6}$$

for all $\tilde{W} = (\tilde{U}, \tilde{U}')$ satisfying the adjoint eigenvalue equation, and vice versa, where

$$\mathcal{S} := \begin{pmatrix} -A & B \\ -B & 0 \end{pmatrix}, \qquad \mathcal{S}^{-1} = \begin{pmatrix} 0 & -B^{-1} \\ B^{-1} & -B^{-1}AB^{-1} \end{pmatrix}. \tag{6.7}$$

Proof. Property (6.6) follows immediately from the relation below, which we obtain from integration by parts:

$$\tilde{W}^* \mathcal{S} W|_{x_1}^{x_2} = \langle (L-\lambda)^* \tilde{U}, U \rangle_{L^2(x_1,x_2)} - \langle \tilde{U}, (L-\lambda) U \rangle_{L^2(x_1,x_2)} = 0,$$

by the definition of the adjoint operator. (Indeed, the righthand side may be viewed as defining the quadratic form $\mathcal{S}$.) □

6.2. **Domain of consistent splitting.** Define

$$\Lambda := \cap \Lambda_j^{\pm}, \quad j = 1, \dots, n, \tag{6.8}$$

where $\Lambda_j^{\pm}$ denote the open sets bounded on the left by the algebraic curves $\lambda_j^{\pm}(\xi)$ determined by the eigenvalues of the symbols $-\xi^2 B_{\pm} - i\xi A_{\pm} + C_{\pm}$ of the limiting constant-coefficient operators

$$L_{\pm} U := B_{\pm} U'' - A_{\pm} U' + C_{\pm} U \tag{6.9}$$

as ξ is varied along the real axis. The curves $\lambda_j^{\pm}(\cdot)$ comprise the essential spectrum of operators $L_{\pm}$. For Majda's model, the computations of Section 4.1 yield

$$\Lambda \subset \Omega_\eta := \{\lambda : \Re\lambda > \max\{-\eta_1 |\Im\lambda|, \eta_2 |\Im\lambda|^2\}\}, \qquad \eta_j > 0. \tag{6.10}$$

Lemma 6.2 ([MaZ3]). *The set Λ is equal to the component containing real $+\infty$ of the domain of consistent splitting (defined in Section 4.1) for the eigenvalue equation of L written as a first-order ODE* (5.1)*.*

6.3. **Basic solution formula.** Let

$$\Phi_j^+ := P_+ V_j^+, \quad j = 1 \dots, n, \tag{6.11}$$

and

$$\Phi_j^- := P_- V_j^-, \quad j = n+1, \dots, 2n \tag{6.12}$$

denote the locally analytic bases of the stable manifold at $+\infty$ and the unstable manifold at $-\infty$ of solutions of the eigenvalue equation (4.6) written as a first-order system (5.1) that were found in Section 5.2 (i.e., the normal modes), and set

$$\Phi^+ := (\Phi_1^+, \dots, \Phi_n^+), \quad \Phi^- := (\Phi_{n+1}^-, \dots, \Phi_{2n}^-), \tag{6.13}$$

and

$$\Phi := (\Phi^+, \Phi^-). \tag{6.14}$$

Define the solution operator from y to x of (4.6), denoted by $\mathcal{F}^{y \to x}$, as

$$\mathcal{F}^{y \to x} = \Phi(x, \lambda)\, \Phi^{-1}(y, \lambda) \tag{6.15}$$

and the projections $\Pi_y^{\pm}$ on the stable manifolds at $\pm\infty$ as

$$\Pi_y^+ = \begin{pmatrix} \Phi^+(y,\lambda) & 0 \end{pmatrix} \Phi^{-1}(y,\lambda) \quad \text{and} \quad \Pi_y^- = \begin{pmatrix} 0 & \Phi^-(y,\lambda) \end{pmatrix} \Phi^{-1}(y,\lambda). \tag{6.16}$$

Then, we have the following general result established in [ZH, MaZ3].

Proposition 6.3 ([ZH, MaZ3]). *With respect to any L^p, $1 \le p \le \infty$, Λ consists entirely of normal points of L, i.e., resolvent points, or isolated eigenvalues of constant multiplicity. On this domain, the resolvent kernel G_λ is meromorphic, with representation*

$$G_\lambda(x,y) = \begin{cases} (I_n,0)\mathcal{F}^{y\to x}\,\Pi_y^+\,\mathcal{S}^{-1}(y)(I_n,0)^{tr} & x>y,\\ -(I_n,0)\mathcal{F}^{y\to x}\,\Pi_y^-\,\mathcal{S}^{-1}(y)(I_n,0)^{tr} & x<y,\end{cases} \tag{6.17}$$

$\mathcal{S}^{-1}$ as described in (6.7). *Moreover, on any compact subset K of $\rho(L)\cap\Lambda$ ($\rho(L)$ denoting resolvent set), there hold the uniform decay estimates*

$$|\partial_x^j\partial_y^k G_\lambda(x,y)| \le Ce^{-\eta|x-y|}, \tag{6.18}$$

$0 \le |j|+|k| \le 1$, *where $C>0$ and $\eta>0$ depend only on K, L.*

Remark 6.4. Formula (6.17) extends [ZH, MaZ3] to the full phase-variable representation

$$\begin{pmatrix} G_\lambda(x,y) & (\partial/\partial y)G_\lambda(x,y)\\ (\partial/\partial x)G_\lambda(x,y) & (\partial/\partial x)(\partial/\partial y)G_\lambda(x,y)\end{pmatrix} = \begin{cases} \mathcal{F}^{y\to x}\,\Pi_y^+\,\mathcal{S}^{-1}(y) & x>y,\\ -\mathcal{F}^{y\to x}\,\Pi_y^-\,\mathcal{S}^{-1}(y) & x<y.\end{cases} \tag{6.19}$$

6.4. **Generalized spectral decomposition.** Formula (6.17) suffices for the description of intermediate- and high-frequency behavior. For the treatment of the key low-frequency regime, it is preferable to use a modified representation of the resolvent kernel consisting of a *scattering decomposition* in solutions of the forward and adjoint eigenvalue equations.

From (6.6), it follows that if there are n independent solutions $\phi_1^+,\dots,\phi_n^+$ of $(L-\lambda I)U=0$ decaying at $+\infty$, and n independent solutions $\phi_{n+1}^-,\dots,\phi_{2n}^-$ of the same equations decaying at $-\infty$, then there exist n independent solutions $\tilde\psi_{n+1}^+,\dots,\tilde\psi_{2n}^+$ of $(L^*-\lambda^* I)\tilde U=0$ decaying at $+\infty$, and n independent solutions $\tilde\psi_1^-,\dots,\tilde\psi_n^-$ decaying at $-\infty$. Precisely, setting

$$\Psi_j^+ := P_+V_j^+, \quad j=n+1\dots,2n, \tag{6.20}$$

$$\Psi_j^- := P_-V_j^-, \quad j=1,\dots,n, \tag{6.21}$$

and

$$\Psi := (\Psi^+,\Psi^-), \tag{6.22}$$

similarly as in (6.11)–(6.14), where $\Psi_j^\pm = (\psi_j^\pm,(\psi_j^\pm)')$ are exponentially growing solutions obtained through Lemma 5.1, we may define dual exponentially decaying and growing solutions $\tilde\psi_j^\pm$ and $\tilde\phi_j^\pm$ via

$$\begin{pmatrix}\tilde\Psi & \tilde\Phi\end{pmatrix}^*\mathcal{S}\begin{pmatrix}\Psi & \Phi\end{pmatrix}_\pm \equiv I. \tag{6.23}$$

Corollary 6.5 ([Z1, MaZ3]). *On $\Lambda\cap\rho(L)$, there hold*

$$G_\lambda(x,y) = \sum_{j,k} M_{jk}^+(\lambda)\phi_j^+(x;\lambda)\tilde\psi_k^-(y;\lambda)^* \tag{6.24}$$

for $y \le 0 \le x$,

$$G_\lambda(x,y) = \sum_{j,k} d_{jk}^+(\lambda)\phi_j^-(x;\lambda)\tilde\psi_k^-(y;\lambda)^* - \sum_k \psi_k^-(x;\lambda)\tilde\psi_k^-(y;\lambda)^* \tag{6.25}$$

for $y \le x \le 0$, and

$$G_\lambda(x,y) = \sum_{j,k} d^-_{jk}(\lambda)\phi^-_j(x;\lambda)\tilde\psi^-_k(y;\lambda)^* + \sum_k \phi^-_k(x;\lambda)\tilde\phi^-_k(y;\lambda)^* \tag{6.26}$$

for $x \le y \le 0$, with

$$M^+ = (-I,0)\begin{pmatrix}\Phi^+ & \Phi^-\end{pmatrix}^{-1}\Psi^- \tag{6.27}$$

and

$$d^\pm = \begin{pmatrix}0, I\end{pmatrix}\begin{pmatrix}\Phi^+ & \Phi^-\end{pmatrix}^{-1}\Psi^-. \tag{6.28}$$

Symmetric representations hold for $y \ge 0$.

Proof. Rearrangement of (6.17) using (6.23); see [MaZ3]. □

Remarks 6.6. 1. This representation reflects the classical duality principle (see, e.g. [ZH], Lemma 4.2) that the transposition $G^*_\lambda(y,x)$ of the Green's function $G_\lambda(x,y)$ associated with operator $(L-\lambda)$ should be the Green's function for the adjoint operator $(L^*-\lambda^*)$.

2. In the constant-coefficient case, with a choice of common bases $\Psi^\pm = \Phi^\mp$ at $\pm\infty$, (6.24)–(6.28) reduce to the simple formula

$$G_\lambda(x,y) = \begin{cases} -\sum_{j=k+1}^N \phi^+_j(x;\lambda)\tilde\phi^{+*}_j(y;\lambda) & x > y,\\ \sum_{j=1}^k \phi^-_j(x;\lambda)\tilde\phi^{-*}_j(y;\lambda) & x < y,\end{cases} \tag{6.29}$$

where, generically, $\phi^\pm_j$, $\tilde\phi^\pm_j$ may be taken as pure exponentials

$$\phi^\pm_j(x)\tilde\phi^{\pm*}_j(y) = e^{\mu^\pm_j(\lambda)(x-y)}V^\pm_j(\lambda)\tilde V^{\pm*}_j(\lambda). \tag{6.30}$$

This reveals an analogy to the usual representation obtained by Fourier transform solution. We see that (6.25) (resp. (6.26)) in the far-field limit $x \to +\infty$ (resp. $x \to -\infty$) consists of the limiting constant-coefficient resolvent kernel plus an exponentially decaying error term.

6.5. **Resolvent kernel bounds.** The basic bound (6.18) is sufficient to treat intermediate frequencies $r \le |\lambda| \le R$, r, $R > 0$ (indeed, it is difficult to say more in this regime). As $|\lambda| \to \infty$, or $\lambda \to 0 \in \partial\rho(L)$, however, the bounds are not uniform, and so separate analyses are needed in these high- and low-frequency regimes.

In the high-frequency ($\sim$ short-time) regime, we have the following classical analytic semigroup-type bounds following from strict parabolicity alone. These may be obtained from the basic formula (6.17) using the parabolic rescaling $x \to x|\lambda|^{1/2}$ and estimates (the Tracking Lemma of [Z1, Z2, MaZ3]) for slowly-varying-coefficient ODE; see, e.g., [Sa, ZH, Z1].

Proposition 6.7 (High-frequency bounds [ZH])**.** *For general strictly parabolic systems of form* (6.3)*, R, $C > 0$ sufficiently large, and η_1, η_2, $\theta > 0$ sufficiently small,*

$$|G_\lambda(x,y)| \le C|\lambda|^{-1/2}e^{-\theta|\lambda|^{\frac12}|x-y|}, \qquad |\partial_x G_\lambda(x,y)|,\ |\partial_y G_\lambda(x,y)| \le Ce^{-\theta|\lambda|^{\frac12}|x-y|} \tag{6.31}$$

for all $\lambda \in \Omega_\eta \setminus B(0,R)$, with Ω_η as defined in (6.10)*.*

Thus, the only resolvent bounds that depend on the details of the model are the crucial low-frequency ($\sim$ large-time) bounds carrying information relevant to large-time asymptotics.

For the Majda model, these are as follows.

Proposition 6.8 (Low-frequency bounds). *Let $\bar U = (\bar u, \bar z)$ be a traveling wave profile of Majda's model, satisfying ($\mathcal{D}$). Then, for $r > 0$ sufficiently small, the resolvent kernel G_λ has a meromorphic extension onto $B(0,r) \subset \mathbb{C}$, which may be decomposed as*

$$G_\lambda = E_\lambda + S_\lambda + R_\lambda, \tag{6.32}$$

where

$$E_\lambda(x,y) := \begin{cases} \lambda^{-1}\bar U'(x)\pi_f^-(y)^{\mathrm{tr}} e^{(\lambda/\alpha^- - \lambda^2/\alpha^{-3})y} & \alpha^- > 0, \\ \lambda^{-1}\bar U'(x)\pi_r^-(y)^{\mathrm{tr}} & \alpha^- < 0, \end{cases} \tag{6.33}$$

for $y \le 0$, with π_j^- bounded solutions of the adjoint eigenvalue equation for $\lambda = 0$, π_f^- convergent as $y \to -\infty$ to cL_f^- and π_r^- exponentially decaying as $y \to -\infty$,

$$E_\lambda(x,y) := \begin{cases} \begin{aligned} &\lambda^{-1}\bar U'(x)\pi_f^+(y)^{\mathrm{tr}} e^{(\lambda/\alpha^+ - \lambda^2/\alpha^{+3})y} \\ &\quad + \lambda^{-1}\bar U'(x)\pi_r^+(y)^{\mathrm{tr}} e^{(-\lambda/s + \lambda^2 d/s^3)y} \end{aligned} & \alpha^+ < 0, \\ \\ \lambda^{-1}\bar U'(x)\pi_r^+(y)^{\mathrm{tr}} e^{(-\lambda/s + \lambda^2 d/s^3)y} & \alpha^+ > 0, \end{cases} \tag{6.34}$$

for $y \ge 0$, with π_j^+ bounded solutions of the adjoint eigenvalue equation for $\lambda = 0$, convergent as $y \to +\infty$;

$$S_\lambda(x,y) := \begin{cases} cR_f^+ L_f^{-t} e^{(-\lambda/\alpha^+ + \lambda^2/\alpha^{+3})x + (\lambda/\alpha^- - \lambda^2/\alpha^{-3})y} & \alpha^-, \alpha^+ > 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.35}$$

for $y \le 0 \le x$, R_f^- and L_f^- constant vectors as defined in (4.24),

$$S_\lambda(x,y) := \begin{cases} R_f^- L_f^{-\mathrm{tr}} e^{(-\lambda/\alpha^- + \lambda^2/\alpha^{-3})(x-y)} & \alpha^- > 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.36}$$

for $y \le x \le 0$, and

$$S_\lambda(x,y) := \begin{cases} R_f^- L_f^{-\mathrm{tr}} e^{(-\lambda/\alpha^- + \lambda^2/\alpha^{-3})(x-y)} & \alpha^- < 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.37}$$

for $x \le y \le 0$, with similar relations for $y \ge 0$; and R_λ denotes a faster-decaying residual term.[4]

Remarks 6.9. 1. Recall, (4.20)–(4.22), (4.24)–(4.25), that vectors $R_f^\pm = (*, 0)^{\mathrm{tr}}$ appearing in scattering terms S_λ have vanishing z-component, and also $L_F \perp (-q, 1)$, a fact that will be important in our later nonlinear stability analysis.

2. The case $\alpha^- < 0$ in (6.33) occurs only in the extreme situation of a strong deflagration, for which there are no incoming characteristics on the lefthand side $y \le 0$. This is essentially the only difference from the corresponding proposition for viscous shock waves in the general Lax or undercompressive case, and represents just an anomaly in bookkeeping.

[4]See, e.g., [MaZ3, Z2] for bounds in the viscous Lax shock $\sim$ strong detonation case.

Proof. Similarly as in the viscous shock case treated in [MaZ3], this follows from representations (6.24)–(6.28) of Corollary 6.5, estimating modes $\phi_j^\pm$, $\psi_j^\pm$, $\tilde\phi_j^\pm$, and $\tilde\psi_j^\pm$ using the asymptotic description given by the Conjugation Lemma together with the constant-coefficient analysis of Section 4.1, and estimating scattering coefficients M_{jk}, $d_{jk}^\pm$ crudely by Laurent series: e.g.,

$$d_{jk} = d_{jk}^{-1}\lambda^{-1} + d_{jk} = d_{jk}^0 + \dots,$$

noting that pole terms of order λ^{-k} correspond to zeroes of order k of the Evans function, hence (by $(\mathcal{D})$) are at most order $k = 1$ and (without loss of generality coordinatizing so that $\phi_1^+ = \phi_{2n}^- = \bar U'(x)$) involve only zero-eigenfunction $\bar U'(x)$ as x-dependent factor. Specifically, E_λ comprises exact pole terms, while S_λ comprises order one terms involving products of slowly decaying forward and dual modes (i.e., modes that are merely bounded for $\lambda = 0$), the latter estimated to exponentially decaying error via the Conjugation Lemma, while R_λ comprises remaining, residual terms.

Vectors R_j, L_j in the formulae for S_λ arise through the limiting, constant-coefficient analysis of Section 4.1. Finally, the information that π_f^- at $\lambda = 0$ converges as $y \to -\infty$ to cL_f^- if $\alpha^- > 0$ and to zero if α^- follows by the fact that in the first case there exists but a single bounded, nondecaying solution of the eigenvalue equation as $y \to -\infty$, with asymptotic direction L_f^-, and in the second case there exists no bounded, nondecaying solution. These facts, in turn, are readily verified by the constant-coefficient analysis of Section 4.1, combined with the Conjugation Lemma. $\square$

6.6. **The system case.** Proposition 6.5 admits a straightforward generalization to the system case (i.e., the vectorial version of (1.3) discussed in Remark 2.1.3), substituting in place of $\alpha^\pm$, $L_f^\pm$, $R_f^\pm$ the eigenvalues $\alpha_j^\pm$ and eigenvectors $L_{f,j}^\pm$, $R_{f,j}^\pm$ of $f_u(U_\pm) - sI$ and in place of L_r^+, R_r^+ the eigenvectors $L_{r,i}^+$, $R_{r,i}^+$ of the (now matrix-valued) diffusion coefficient d, as described in Section 4.2 (recall, there are no slow reactive modes on the minus infinity side $x \le 0$). See Proposition 4.22, [Z2] for a corresponding description in the viscous shock case.

Proposition 6.10 (Low-frequency bounds)**.** *Let $\bar U = (\bar u, \bar z)$ be a traveling wave profile of vectorial Majda's model under dissipativity hypothesis* (3.15)*, satisfying $(\mathcal{D})$.*[5] *Then, for $r > 0$ sufficiently small, the resolvent kernel G_λ has a meromorphic extension onto $B(0,r) \subset \mathbb{C}$, which may be decomposed as*

$$G_\lambda = E_\lambda + S_\lambda + R_\lambda, \tag{6.38}$$

where

$$E_\lambda(x,y) := \begin{cases} \lambda^{-1}\sum_{\alpha_j^->0} \bar U'(x)\pi_{f,j}^-(y)^{\mathrm{tr}} e^{(\lambda/\alpha_j^- - \lambda^2 b_j^-/\alpha_j^{-3})y} & \text{some } \alpha_j^- > 0,\\ \lambda^{-1}\bar U'(x)\pi_r^-(y)^{\mathrm{tr}} & \text{all } \alpha_j^- < 0, \end{cases} \tag{6.39}$$

for $y \le 0$, with $\pi_{f,j}^-$ bounded solutions of the adjoint eigenvalue equation for $\lambda = 0$, exponentially convergent as $y \to -\infty$ to $c_j^- L_{f,j}^-$ and $\pi_{r,i}^-$ exponentially decaying as $y \to -\infty$,

[5]Recall, Remark (1.7), this implies in part that $\bar U$ is a unique, transversal connection.

$$
(6.40) \qquad E_\lambda(x,y) := \begin{cases} \lambda^{-1} \sum_{\alpha_j^+<0} \bar U'(x)\pi_{f,j}^+(y)^{\rm tr} e^{(\lambda/\alpha_j^+ - \lambda^2 b_j^+/\alpha_j^{+3})y} & \\ \quad + \lambda^{-1} \sum_{i=1}^m \bar U'(x)\pi_{r,i}^+(y)^{\rm tr} e^{(-\lambda/s+\lambda^2 d_i^+/s^3)y} & \text{some } \alpha_j^+<0, \\ & \\ \lambda^{-1} \sum_{i=1}^m \bar U'(x)\pi_{r,i}^+(y)^{\rm tr} e^{(-\lambda/s+\lambda^2 d_i^+/s^3)y} & \text{all } \alpha_j^+>0, \end{cases}
$$

for $y \ge 0$, *with* $\pi_{f,j}^+$, π_r^+ *bounded solutions of the adjoint eigenvalue equation for* $\lambda = 0$, *exponentially convergent as* $y \to +\infty$*;*

$$
(6.41) \qquad S_\lambda(x,y) := \sum_{\alpha_k^-,\alpha_j^+>0} c_{k,-}^{j,+} R_{f,j}^+ {L_{f,k}^-}^t e^{(-\lambda/\alpha_j^+ + \lambda^2 b_j^+/\alpha_j^{+3})x + (\lambda/\alpha_j^- - \lambda^2 b_k^-/\alpha_j^{-3})y}
$$

for $y \le 0 \le x$, $R_{f,j}^-$ *and* $L_{f,k}^-$ *constant vectors as defined in Section 4.2,*

$$
(6.42) \qquad \begin{aligned} S_\lambda(x,y) := & \sum_{a_k^->0} R_k^- {L_k^-}^t e^{(-\lambda/\alpha_k^- + \lambda^2 b_k^-/\alpha_k^{-3})(x-y)} \\ & + \sum_{\alpha_{f,j}^-<0, \alpha_k^->0} c_{k,-}^{j,-} R_{f,j}^- {L_{f,k}^-}^{\rm tr} e^{(-\lambda/\alpha_j^- + \lambda^2 b_j^-/\alpha_j^{-3})x + (\lambda/\alpha_k^- - \lambda^2 b_k^-/\alpha_k^{-3})y} \end{aligned}
$$

for $y \le x \le 0$, *and*

$$
(6.43) \qquad \begin{aligned} S_\lambda(x,y) := & \sum_{\alpha_k^-<0} R_k^- {L_k^-}^t e^{(-\lambda/\alpha_k^- + \lambda^2 b_k^-/\alpha_k^{-3})(x-y)} \\ & + \sum_{\alpha_{f,j}^-<0, \alpha_k^->0} c_{k,-}^{j,-} R_{f,j}^- {L_{f,k}^-}^{\rm tr} e^{(-\lambda/\alpha_j^- + \lambda^2 b_j^-/\alpha_j^{-3})x + (\lambda/\alpha_k^- - \lambda^2 b_k^-/\alpha_k^{-3})y} \end{aligned}
$$

for $x \le y \le 0$, *where* $c_j^\pm$, $c_{k,\pm}^{j,\pm}$ *are scalar constants, with similar relations for* $y \ge 0$*; and* R_λ *denotes a faster-decaying residual term.*

7. Green function bounds

We may now estimate the Green function $G(x,t;y) := e^{Lt}\delta_y(x)$ associated with the linearized operator L about the wave, determined by

$$
(7.1) \qquad (\partial_t - L)G = 0, \qquad G(x,0;y) = \delta_y(x),
$$

via the inverse Laplace-transform formula, following the approach of [ZH, MaZ3]. We present our results using a bookkeeping scheme similar to that of [HZ] in the undercompressive viscous shock case.

7.1. **Basic bounds.** Recall the standard notation

$$\mathrm{errfn}(z) := \frac{1}{2\pi}\int_{-\infty}^{z} e^{-\xi^2} d\xi.$$

Proposition 7.1. *Let $\bar U = (\bar u, \bar z)$ be a traveling wave profile of Majda's model, satisfying $(\mathcal{D})$. Then, the Green function $G(x,t;y)$ associated with the linearized equations (6.3) may be decomposed as $G = E + \tilde G$, where*

$$E(x,t;y) = \bar U'(x) e(y,t), \tag{7.2}$$

$$e(y,t) = \begin{cases} \left(errfn\ \left(\frac{y+\alpha^- t}{\sqrt{4t}}\right) - errfn\ \left(\frac{y-\alpha^- t}{\sqrt{4t}}\right)\right)\pi_f^-(y) & \alpha^- > 0, \\ \pi_r^-(y) & \alpha^- < 0, \end{cases} \tag{7.3}$$

for $y \le 0$ and

$$e(y,t) := \begin{cases} \left(errfn\ \left(\frac{y+\alpha^+ t}{\sqrt{4t}}\right) - errfn\ \left(\frac{y+\alpha^+ t}{\sqrt{4t}}\right)\right)\pi_f^+(y) & \\ \quad + \left(errfn\ \left(\frac{y+st}{\sqrt{4dt}}\right) - errfn\ \left(\frac{y-st}{\sqrt{4dt}}\right)\right)\pi_r^+(y) & \alpha_+ < 0, \\ \left(errfn\ \left(\frac{y+st}{\sqrt{4dt}}\right) - errfn\ \left(\frac{y-st}{\sqrt{4dt}}\right)\right)\pi_r^+(y) & \alpha^+ > 0, \end{cases} \tag{7.4}$$

for $y \ge 0$, with $\pi_j^\pm$ as in Proposition 6.8: in particular,

$$|\pi_j^\pm| \le C, \qquad |\partial_y \pi_j^\pm| \le C e^{-\eta|y|}, \tag{7.5}$$

with $|\pi_f^-| \le Ce^{-\eta|y|}$ if $\alpha^- < 0$ (strong deflagration case) and $\pi_f^\pm \to L_f^\pm$ as $x \to -\infty$ otherwise, and, denoting by $a_j^\pm$ the "undamped" characteristic speeds $a_f^- = \alpha^-$, $a_f^+ = \alpha^+$, and $a_r^+ = -s$ (Note: a_r^- does not appear),

$$\begin{aligned} |\partial_{x,y}^\alpha \tilde G(x,t;y)| &\le Ce^{-\eta(|x-y|+t)} \\ &+ \quad C(t^{-|\alpha|/2} + |\alpha_x| e^{-\eta|x|} + |\alpha_y| e^{-\eta|y|}) \\ &\quad \times \Big(\sum_k t^{-1/2} e^{-(x-y-a_k^- t)^2/Mt} e^{-\eta x^+} \\ &+ \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t| \ge |y|\}} t^{-1/2} e^{-(x - a_j^-(t-|y/a_k^-|))^2/Mt} e^{-\eta x^+} \\ &+ \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t| \ge |y|\}} t^{-1/2} e^{-(x - a_j^+(t-|y/a_k^-|))^2/Mt} e^{-\eta x^-} \\ &+ \sum_{a_k^->0} \chi_{\{|a_k^- t| \ge |y|\}} t^{-1/2} e^{-(x+s(t-|y/a_k^-|))^2/Mt} e^{-\eta|x|} \Big), \end{aligned} \tag{7.6}$$

$0 \le |\alpha| \le 1$ for $y \le 0$ and symmetrically for $y \ge 0$, for some η, C, $M > 0$, where $x^\pm$ denotes the positive/negative part of x and indicator function $\chi_{\{|a_k^- t| \ge |y|\}}$ is 1 for $|a_k^- t| \ge |y|$ and 0 otherwise. Moreover, for $x \le 0$, $|(0,1)\tilde G(x,t;y)|$ decays at the faster x-derivative rate

$\alpha_x = 1$, *as does* $|(0,1)\tilde{G}(x,t;y)(1,0)^{\rm tr}|$ *for any* x, *and, for* $y \le 0$, $|\tilde{G}(x,t;y)(-q,1)^{\rm tr}|$ *decays at the faster* y*-derivative rate* $\alpha_y = 1$.

Proof. Reflecting the formal relation that G_λ is Laplace transform of G, we have the Inverse Laplace transform formula

$$G(x,t;y) = \frac{1}{2\pi i} \oint_\Gamma e^{\lambda t} G_\lambda(x,y)\, d\lambda, \tag{7.7}$$

where $\Gamma = \partial\{\lambda : Re\,\lambda > \theta_1 - \theta_2 |\Im \lambda|\}$ is the boundary of an appropriate sector containing the spectrum of L, $\theta_2 > 0$. Following [ZH, MaZ3], we may thus convert the detailed resolvent kernel estimates of Proposition 6.5 to estimates on the Green function via stationary phase, or Riemann saddlepoint, estimates on (7.7), exactly as was done in the viscous shock case.

Specifically, using the property that G_λ is meromorphic on $\Omega_\eta \cup B(0,r)$ for r, $\eta > 0$ sufficiently small (see Propositions 6.3, 6.7, and 6.8) and analytic on the the resolvent set $\rho(L)$, we may estimate the contribution of each of the various meromorphic components C_λ of G_λ by a combination of direct evaluation using Calculus of residues and strategic deformation of the contour Γ so as to minimize

$$\oint_\Gamma |e^{\lambda t} C_\lambda(x,y)|\, |d\lambda| = \oint_\Gamma e^{\Re \lambda t} |C_\lambda(x,y)|\, |d\lambda|$$

for each fixed x, y, t, with the main contribution to E coming from explicit evaluation of the corresponding low-frequency term E_λ in Propositon 6.8 and the main contribution to $\tilde{G}$ coming from explicit evalution of S_λ. See [ZH, MaZ3, Z2] for details.

It remains only to verify the key properties of faster decay of $|(0,1)\tilde{G}(x,t;y)|$ for $x \le 0$, $|(0,1)\tilde{G}(x,t;y)(1,0)^{\rm tr}|$ for general x, and $|\tilde{G}(x,t;y)(-q,1)^{\rm tr}|$ for $y \le 0$. To see the first property, we have only to observe, in the bounds of Proposition 6.5, that, for $x \le 0$, only fluid modes appear in the rate-determining term S_λ, and these lie in direction $R_f^- = (*,0)^{\rm tr}$ having vanishing z-component.[6] For $x \ge 0$, fluid modes again lie in direction $R_f^+ = (*,0)^{\rm tr}$ having vanishing z-component, while reactive terms appear as scalar multiples of projector $R_r^+(L_r^+)^{\rm tr} = (0,*)$ orthogonal to $(1,0)^{\rm tr}$. Thus, $|(0,1)\tilde{G}(x,t;y)(1,0)^{\rm tr}|$ to lowest order involves only z-components of fluid terms, hence again is faster decaying; this yields the second property. Likewise, for $y \le 0$, $S_\lambda(-q,1)^{\rm tr} = 0$, since $L_f^- \perp (-q,1)^{\rm tr}$, and this yields the third property, completing the proof. $\square$

Remarks 7.2. 1. Similarly as in the viscous or relaxation shock case, the bounds of Proposition 7.1 may be interpreted as describing the evlution of an initial delta-function perturbation at y as the superposition of signals convecting along hyperbolic characteristics and diffusing as approximate Gaussians until they strike the shock layer, whereupon they scatter as reflected and transmitted waves along outgoing characteristics, at the same time exciting the stationary mode $\bar{U}'$. The main new feature in the reacting as compared to the nonreacting case is the exiting signal along the reaction characteristic on the lefthand ($x \le 0$) side, for which a constant-coefficient analysis indicates that the Gaussian signal is now exponentially decaying in time, due to burning of the reactant. This is reflected in the final term of (7.6), consisting of an ordinary Gaussian reflected left into an exponentially penalized field $e^{-\theta|x|}$, a term indistinguishable in modulus bound from a Gaussian multiplied by a factor

[6] The restriction $x \le 0$ is necessary because of incoming (i.e., leftmoving) undamped reaction waves for the case $y \ge 0$ not listed in Proposition 7.1.

decaying exponentially in the travel time after reflection. We call this leftgoing reactive characteristic speed “damped” and all others “undamped” to distinguish this behavior.

2. A second difference between the reacting and nonreacting case, this time confined to (scalar) Majda's model, is the different structure of the excited term E in the case $\alpha^- < 0$, a book-keeping anomaly arising because of the absence of incoming waves on the lefthand side in this (strong deflagration) case. This different structure has essentially no effect on the analysis; see Remark 8.5.

3. The improved bounds for $|(0,1)\tilde G(x,t;y)|$ and $|\tilde G(x,t;y)(-q,1)^{\mathrm{tr}}|$ for $x \le 0$ are similar to those of the relaxation case [MaZ1], to which the linearized equations are analogous on the side $x \le 0$ on which $\phi > 0$, and play a similarly important role in the later nonlinear stability analysis. In conservative coordinates $w := u + qz$, z of (2.2)–(2.3), these bounds have the simpler statement that the Green function decays more rapidly in its z-components, both output and input.

7.2. **The system case.** The somewhat cumbersome summation notation of Proposition 7.1 is designed for easy generalization to the system case. Indeed, starting from Remark 6.9.3, it is straightforward to verify the analogous theorem for the full reactive Navier–Stokes equations with artificial viscosity– more generally, the abstract vectorial model $u \in \mathbb{R}^n$, $z \in \mathbb{R}^m$ described in Remark 2.1.3– with undamped characteristic modes now $a_j^\pm = \alpha_1^\pm, \dots, \alpha_n^\pm$, $-s$ ($-s$ multiplicity $m = \dim z$), $\alpha_j^\pm$ and $L_f^\pm$ denoting the eigenvalues and left eigenvectors of $(\partial f/\partial u)(U_\pm) - sI$. See [HZ] for the analogous description in the viscous shock case.

Proposition 7.3. *Let $\bar U = (\bar u, \bar z)$ be a traveling wave profile of vectorial Majda's model under dissipativity hypothesis* (3.15)*, satisfying ($\mathcal{D}$).*[7] *Then, the Green function $G(x,t;y)$ associated with the linearized equations* (6.3) *may be decomposed as $G = E + \tilde G$, where*

$$E(x,t;y) = \bar U'(x) e(y,t), \tag{7.8}$$

$$e(y,t) = \begin{cases} \sum_{\alpha_j^- > 0} \left(errfn \left(\frac{y + \alpha_j^- t}{\sqrt{4 b_j^- t}} \right) - errfn \left(\frac{y - \alpha_j^- t}{\sqrt{4 b_j^- t}} \right) \right) \pi_{f,j}^-(y)^{\mathrm{tr}} & \text{some } \alpha_j^- > 0, \\ \pi_r^-(y) & \text{all } \alpha_j^- < 0, \end{cases} \tag{7.9}$$

for $y \le 0$ and

$$e(y,t) := \begin{cases} \displaystyle\sum_{\alpha_j^+ < 0} \left(errfn \left(\frac{y + \alpha_j^+ t}{\sqrt{4 b_j^+ t}} \right) - errfn \left(\frac{y + \alpha_j^+ t}{\sqrt{4 b_j^+ t}} \right) \right) \pi_{f,j}^+(y)^{\mathrm{tr}} & \\ \quad + \displaystyle\sum_{i=1}^{m} \left(errfn \left(\frac{y + st}{\sqrt{4 d_i^+ t}} \right) - errfn \left(\frac{y - st}{\sqrt{4 d_i^+ t}} \right) \right) \pi_{r,i}^+(y)^{\mathrm{tr}} & \text{some } \alpha_j^+ < 0, \\ \sum_{i=1}^{m} \left(errfn \left(\frac{y + st}{\sqrt{4 d_i^+ t}} \right) - errfn \left(\frac{y - st}{\sqrt{4 d_i^+ t}} \right) \right) \pi_{r,i}^+(y)^{\mathrm{tr}} & \text{all } \alpha_j^+ > 0, \end{cases} \tag{7.10}$$

[7] Recall, (1.7), this implies in part that $\bar U$ is a unique, transversal connection.

for $y \geq 0$, with $\pi_j^\pm$ as in Proposition 6.8: in particular,

$$|\pi_j^\pm| \leq C, \qquad |\partial_y \pi_j^\pm| \leq Ce^{-\eta|y|}, \tag{7.11}$$

with $|\pi_f^-| \leq Ce^{-\eta|y|}$ if $\alpha^- < 0$ (strong deflagration case) and $\pi_f^\pm \to L_f^\pm$ as $x \to -\infty$ otherwise, and $\tilde{G}$ satisfies (7.6). Moreover, for $x \leq 0$, $|(0, I_r)\tilde{G}(x,t;y)|$ decays at the faster x-derivative rate $\alpha_x = 1$, as does $|(0, I_r)\tilde{G}(x,t;y)(I_n, 0)^{\mathrm{tr}}|$ for any x, and, for $y \leq 0$, $|\tilde{G}(x,t;y)(-q^{\mathrm{tr}}, I_r)^{\mathrm{tr}}|$ decays at the faster y-derivative rate $\alpha_y = 1$.

7.3. **Linearized stability criterion.**

Proof of Theorem 1.2. Sufficiency of $(\mathcal{D})$ for linearized orbital stability follows immediately by the bounds of Theorem 7.1 (resp. Remark 7.2.4) and standard $L^q \to L^p$ convolution bounds, exactly as in the viscous shock case, setting

$$\delta(t) := \int_{-\infty}^{+\infty} E(x,t;y)u_0(y)dy$$

so that

$$U - \delta(t)\bar{U}' = \int_{-\infty}^{+\infty} \tilde{G}(x,t;y)u_0(y)dy;$$

see [ZH, MaZ3, Z2] for further details. Necessity follows from more general spectral considerations not requiring the detailed bounds of Theorem 7.1; see the discussion of effective spectrum in [ZH, MaZ3, Z2]. The argument goes again exactly as in the viscous shock case. □

8. Nonlinear stability

We can now readily establish nonlinear stability by a combination of the methods used in [HZ] to treat general undercompressive viscous shock waves ($\sim x \geq 0$ behavior), and the methods used in [MaZ1] to treat relaxation shocks ($\sim x \leq 0$ behavior). As it costs no additional effort in bookkeeping, we carry out this part of the argument in the full generality of the system case. Recall (Remark 2.1.3) that this includes the artificial viscosity version of the full reactive Navier–Stokes equations with multi-species reaction and reaction-dependent equation of state.

Denoting by $a_j^\pm$ the "undamped" characteristic speeds: $a_f^- = \alpha^-$, $a_f^+ = \alpha^+$, and $a_r^+ = -s$ for Majda's model; $a_{f,i}^- = \alpha_i^-$, $i = 1, \dots, n$, $a_{f,i}^+ = \alpha_i^+$, $i = 1, \dots, n$, $a_{r,i}^+ = -s$, $i = 1, \dots, m$ in the system case $u \in \mathbb{R}^n$, $z \in \mathbb{R}^m$; define

$$\theta(x,t) := \sum_{a_j^- < 0} (1+t)^{-1/2} e^{-|x - a_j^- t|^2/Lt} + \sum_{a_j^+ > 0} (1+t)^{-1/2} e^{-|x - a_j^+ t|^2/Lt}, \tag{8.1}$$

$$\begin{aligned} \psi_1(x,t) := \chi(x,t) &\sum_{a_j^- < 0} (1+|x|+t)^{-1/2}(1+|x - a_j^- t|)^{-1/2} \\ &+ \chi(x,t) \sum_{a_j^+ > 0} (1+|x|+t)^{-1/2}(1+|x - a_j^+ t|)^{-1/2}, \end{aligned} \tag{8.2}$$

and

$$\psi_2(x,t) := (1-\chi(x,t))(1+|x-a_1^- t|+t^{1/2})^{-3/2} + (1-\chi(x,t))(1+|x-a_n^+ t|+t^{1/2})^{-3/2}, \tag{8.3}$$

where $L>0$ is a sufficiently large constant and $\chi(x,t)=1$ for

$$x \in [\min_j\{a_j^- t, 0\}, \max_j\{a_j^+ t, 0\}], \tag{8.4}$$

that is, for x between the extremal outgoing undamped characteristics, and zero otherwise.[8]

Then, we have the following pointwise version of Theorem 1.5.

Proposition 8.1. *Let $\bar U(x-st)$ be a traveling combustion wave of Majda's model (more generally, the vectorial version including reactive Navier–Stokes equations with artificial viscosity) and $|U_0(x)| \le E_0(1+|x|)^{-3/2}$, E_0 sufficiently small. Then, there exist $\delta(\cdot)$ and $\delta(+\infty)$ such that*

$$\begin{aligned} |\tilde U(x,t) - \bar U^{\delta(t)}(x)| &\le CE_0(\theta+\psi_1+\psi_2)(x,t),\\ |\dot\delta(t)| &\le CE_0(1+t)^{-1},\\ |\delta(t)-\delta(+\infty)| &\le CE_0(1+t)^{-1/2}, \end{aligned} \tag{8.5}$$

where $\tilde U$ denotes the solution of the same equations with perturbed initial data $\tilde U_0 = \bar U + U_0$.

As discussed in the introduction, we establish Proposition 8.1 by a combination of the analysis of undercompressive viscous shock waves in [HZ] and of relaxation shocks in [MaZ1].

Following [HZ], set

$$U(x,t) := \tilde U(x+\delta(t),t) - \bar U(x), \tag{8.6}$$

so that (6.1) becomes by Taylor expansion of F, G:

$$U_t - LU = Q(U)_x + R(U) + \dot\delta(t)(\bar U_x + U_x), \tag{8.7}$$

L as in (6.3), where $R(U) = (-q,1)^{\mathrm{tr}} r(U)$, $r(U)$ scalar, with

$$\begin{aligned} Q(U) &= \mathcal{O}(|U|^2),\\ r(U) &= \mathcal{O}(|U|^2 e^{-\eta x^+}), \end{aligned} \tag{8.8}$$

so long as $|U|$ remains bounded, where x^+ denotes the positive part of x and $\eta>0$.

Remark 8.2. Here, in the description of R, r we have used the specific form

$$G(U) = -\phi(u)\begin{pmatrix} -qk \\ k \end{pmatrix} \tag{8.9}$$

of the reactive source in (6.1), together with Taylor expansion

$$(\phi(\bar u+u)(\bar z+z) - (\phi(\bar u)(\bar z) - (\phi'(\bar u)u\bar z + \phi(\bar u)z) = \phi'(\bar u)uz + \phi''(\bar u+\theta u)u^2\bar z, \tag{8.10}$$

$0<\beta<1$, and the fact that $\phi'(\bar u + v) \le Ce^{-\eta x^+}$ for $|v|$ sufficiently small, by assumption $u_+ \notin [u_i, u^i]$, the property that $\phi'(u)\equiv 0$ for $u \notin [u_i,u^i]$, and exponential convergence of $\bar U(x)$ to U_+ as $x\to+\infty$. This computation, and its exploitation in the later argument (see

[8] This repairs a minor omission in [HZ], where (8.4) was stated incorrectly as $x \in [\min_j\{a_j^- t\}, \max_j\{a_j^+ t\}]$. The formulae differ in the case that there are no outgoing characteristics on one side: extreme Lax shock or strong detonation.

especially the auxiliary bounds of Lemma 8.6), are the main new features in the combustion context as compared to the undercompressive viscous shock wave case.

Recalling the standard fact that $\bar U'$ is a stationary solution of the linearized equations (6.3), $L\bar U' = 0$, or

$$\int_{-\infty}^{\infty} G(x,t;y)\bar U_x(y)dy = e^{Lt}\bar U_x(x) = \bar U'(x),$$

we have by Duhamel's principle:

$$\begin{aligned} U(x,t) = &\int_{-\infty}^{\infty} G(x,t;y)U_0(y)\,dy \\ &+ \int_0^t \int_{-\infty}^{\infty} G(x,t-s;y)(-q,1)^{\mathrm{tr}} r(U)(y,s)\,dy\,ds \\ &- \int_0^t \int_{-\infty}^{\infty} G_y(x,t-s;y)(Q(U)+\dot\delta U)(y,s)\,dy\,ds + \delta(t)\bar U'(x). \end{aligned}$$

Defining

$$\begin{aligned} \delta(t) = &-\int_{-\infty}^{\infty} e(y,t)U_0(y)\,dy \\ &- \int_0^t \int_{-\infty}^{\infty} e(y,t-s)(-q,1)^{\mathrm{tr}} r(U)(y,s)\,dy\,ds \\ &+ \int_0^t \int_{-\infty}^{+\infty} e_y(y,t-s)(Q(U)+\dot\delta\, U)(y,s)dyds, \end{aligned} \tag{8.11}$$

following [Z3, MaZ1, MaZ2, MaZ4], where e is defined as in (7.3)–(7.4) (that is, $e = \sum_j e_j$), and recalling the decomposition $G = E + \tilde G$, we obtain finally the *reduced equations*:

$$\begin{aligned} U(x,t) = &\int_{-\infty}^{\infty} \tilde G(x,t;y)U_0(y)\,dy \\ &+ \int_0^t \int_{-\infty}^{\infty} \tilde G(x,t-s;y)(-q,1)^{\mathrm{tr}} r(U)(y,s)\,dy\,ds \\ &- \int_0^t \int_{-\infty}^{\infty} \tilde G_y(x,t-s;y)(Q(U)+\dot\delta U)(y,s)dy\,ds, \end{aligned} \tag{8.12}$$

and, differentiating (8.11) with respect to t, and observing that $e_y(y,s) \rightharpoonup 0$ as $s \to 0$, as the difference of approaching heat kernels:

$$\begin{aligned} \dot\delta(t) = &-\int_{-\infty}^{\infty} e_t(y,t)U_0(y)\,dy \\ &+ \int_0^t \int_{-\infty}^{\infty} e_t(y,t-s)(-q,1)^{\mathrm{tr}} r(U)(y,s)\,dy\,ds \\ &+ \int_0^t \int_{-\infty}^{+\infty} e_{yt}(y,t-s)(Q(U)+\dot\delta U)(y,s)\,dy\,ds. \end{aligned} \tag{8.13}$$

The following integral estimates are established in [HZ].

Lemma 8.3 (Linear estimates [HZ]). *Under the assumptions of Theorem 1.5,*

$$\begin{aligned}\int_{-\infty}^{+\infty} |\tilde{G}(x,t;y)|(1+|y|)^{-3/2}\,dy \le C(\theta+\psi_1+\psi_2)(x,t),\\ \int_{-\infty}^{+\infty} |e_t(y,t)|(1+|y|)^{-3/2}\,dy \le C(1+t)^{-3/2},\\ \int_{-\infty}^{+\infty} |e(y,t)|(1+|y|)^{-3/2}\,dy \le C,\\ \int_{-\infty}^{+\infty} |e(y,t)-e(y,+\infty)|(1+|y|)^{-3/2}\,dy \le C(1+t)^{-1/2},\end{aligned} \tag{8.14}$$

for $0 \le t \le +\infty$, some $C>0$, where $\tilde{G}$ and e are defined as in Proposition 7.1.

Lemma 8.4 (Nonlinear estimates [HZ]). *Under the assumptions of Theorem 1.5,*

$$\begin{aligned}\int_0^t\int_{-\infty}^{+\infty} |\tilde{G}_y(x,t-s;y)|\Psi(y,s)\,dyds \le C(\theta+\psi_1+\psi_2)(x,t),\\ \int_0^t\int_{-\infty}^{+\infty} |e_{yt}(y,t-s)|\Psi(y,s)\,dyds \le C(1+t)^{-1},\\ \int_t^{+\infty}\int_{-\infty}^{+\infty} |e_y(y,+\infty)|\Psi(y,s)\,dy \le C\gamma(1+t)^{-1/2},\\ \int_0^t\int_{-\infty}^{+\infty} |e_y(y,t-s)-e_y(y,+\infty)|\Psi(y,s)\,dyds \le C(1+t)^{-1/2},\end{aligned} \tag{8.15}$$

for $0 \le t \le +\infty$, some $C>0$, where $\tilde{G}$ and e are defined as in Proposition 7.1 and

$$\begin{aligned}\Psi(y,s) := (1+s)^{1/2}s^{-1/2}(\theta+\psi_1+\psi_2)^2(y,s)\\ + (1+s)^{-1}(\theta+\psi_1+\psi_2)(y,s).\end{aligned} \tag{8.16}$$

Remark 8.5. The case $\sigma(\alpha^-) < 0$ in (7.3), (7.9), occurring for strong deflagrations as described in Remark 6.9.2, is the only one requiring discussion, since in all other cases the bounds are identical to those of the shock case. We have only to note that $|e|$, $|e_y|$, $|e_{yt}| = |e_t| \equiv 0$ in this case also satisfies the same bounds (or better) that are actually used in the proofs of Lemmas 8.3 and 8.4.

To these, we add the following auxiliary estimates special to the combustion case.

Lemma 8.6 (Auxiliary estimates)**.** *Under the assumptions of Theorem 1.5,*

(8.17)

$$\int_0^t \int_{-\infty}^{+\infty} |\tilde G(x,t-s;y)(-q,1)^{\mathrm{tr}} e^{-\eta y^+}|\Psi(y,s)\,dyds \le C(\theta+\psi_1+\psi_2)(x,t),$$

$$\int_0^t \int_{-\infty}^{+\infty} |e_t(y,t-s)(-q,1)^{\mathrm{tr}} e^{-\eta y^+}|\Psi(y,s)\,dyds \le C(1+t)^{-1},$$

$$\int_t^{+\infty} \int_{-\infty}^{+\infty} |e(y,+\infty)(-q,1)^{\mathrm{tr}} e^{-\eta y^+}|\Psi(y,s)\,dy \le C\gamma(1+t)^{-1/2},$$

$$\int_0^t \int_{-\infty}^{+\infty} |\big(e(y,t-s)-e(y,+\infty)\big)(-q,1)^{\mathrm{tr}} e^{-\eta y^+}|\Psi(y,s)\,dyds \le C(1+t)^{-1/2},$$

for $0\le t\le +\infty$*, some* $C>0$*,* $\tilde G$ *and* e *as in Proposition 7.1 and* Ψ *as in* (8.16).

Proof. For the $\tilde G$-estimate, we have only to recall that, by the bounds of Proposition 7.1, $\tilde G(-q,1)^{\mathrm{tr}}$ obeys the bounds of $\tilde G_y$ for $y\le 0$, while $\tilde G e^{-\eta y^+}$ obeys the bounds of $\tilde G_y$ for $y\ge 0$. The e-estimates follow similarly, by the observation that, for $y\le 0$, $e(-q,1)^{\mathrm{tr}}$ and $e_t(-q,1)^{\mathrm{tr}}$ decay like $(e^{-\eta|y|}+(1+t)^{-1/2})$ times the bounds for e and e_t, hence obey the bounds for $|e_y|$ and e_{yt}, since π_f^- is asymptotically parallel to $L_f^-\perp(-q,1)^{\mathrm{tr}}$, with convergence at exponential rate. Likewise, for $y\ge 0$, $ee^{-\eta y^+}$ and $e_te^{-\eta y^+}$ decay like $(e^{-\eta|y|}+(1+t)^{-1/2})$ times the bounds for e and e_t, hence obey the bounds for $|e_y|$ anb $|e_{yt}$. Thus, all bounds follow by the same arguments as in the proof of Lemma 8.4. □

Proof of Proposition 8.1. With these observations, the proof of nonlinear stability goes essentially as in [HZ]. Define

$$\zeta(t) := \sup_{y,0\le s\le t} \Big(|U|(\theta+\psi_1+\psi_2)^{-1}(y,t) + |\dot\delta(s)|(1+s)\Big). \tag{8.18}$$

We shall establish:

Claim. For all $t\ge 0$ for which a solution exists with ζ uniformly bounded by some fixed, sufficiently small constant, there holds

$$\zeta(t)\le C_2(E_0+\zeta(t)^2). \tag{8.19}$$

From this result, provided $E_0<1/4C_2^2$, we have that $\zeta(t)\le 2C_2E_0$ implies $\zeta(t)<2C_2E_0$, and so we may conclude by continuous induction that

$$\zeta(t)<2C_2E_0 \tag{8.20}$$

for all $t\ge 0$. (By standard short-time existence for artificial viscosity systems (see, e.g., [HoS, ZH]), $U\in C^1$ exists and ζ remains continuous so long as ζ remains bounded by some uniform constant, hence (8.20) is an open condition.) Thus, it remains only to establish the claim above.

Proof of Claim. We must show that $U(\theta+\psi_1+\psi_2)^{-1}$ and $|\dot\delta(s)|(1+s)$ are each bounded by $C(E_0+\zeta(t)^2)$, for some $C>0$, all $0\le s\le t$, so long as ζ remains sufficiently small.

Recalling definition (8.18), we obtain for all $t \geq 0$ and some $C > 0$ that

$$\begin{aligned} |\dot{\delta}(t)| &\leq \zeta(t)(1+t)^{-1}, \\ |U(x,t)| &\leq \zeta(t)(\theta + \psi_1 + \psi_2)(x,t), \end{aligned} \tag{8.21}$$

and therefore

$$|(Q(U) + \dot{\delta} U)(y,s)| \leq C\zeta(t)^2 \Psi(y,s) \tag{8.22}$$

with Ψ as defined in (8.16), for $0 \leq s \leq t$.

Combining (8.22) with representations (8.12)–(8.13) and applying Lemmas 8.3 and 8.4, we obtain

$$\begin{aligned} |U(x,t)| \leq & \int_{-\infty}^{\infty} |\tilde{G}(x,t;y)||U_0(y)|\, dy \\ & + \int_0^t \int_{-\infty}^{\infty} |e(y,t-s)(-q,1)^{\mathrm{tr}}||r(U)(y,s)|\, dy\, ds \\ & + \int_0^t \int_{-\infty}^{\infty} |\tilde{G}_y(x,t-s;y)||(Q(U)+\dot{\delta}U)(y,s)| dy\, ds \\ \leq & \, E_0 \int_{-\infty}^{\infty} |\tilde{G}(x,t;y)|(1+|y|)^{-3/2}\, dy \\ & C\zeta(t)^2 \int_0^t \int_{-\infty}^{\infty} |e(y,t-s)(-q,1)^{\mathrm{tr}} e^{-\eta y^+}|\Psi(y,s)|\, dy\, ds \\ & + C\zeta(t)^2 \int_0^t \int_{-\infty}^{\infty} |\tilde{G}_y(x,t-s;y)|\Psi(y,s) dy\, ds \\ \leq & \, C(E_0 + \zeta(t)^2)(\theta + \psi_1 + \psi_2)(x,t) \end{aligned}$$

and, similarly,

$$\begin{aligned} |\dot{\delta}(t)| \leq & \int_{-\infty}^{\infty} |e_t(y,t)||U_0(y)|\, dy \\ & + \int_0^t \int_{-\infty}^{\infty} |e_t(y,t-s)(-q,1)^{\mathrm{tr}}||r(U)(y,s)|\, dy\, ds \\ & + \int_0^t \int_{-\infty}^{+\infty} |e_{yt}(y,t-s)||(Q(U)+\dot{\delta}U)(y,s)|\, dy\, ds \\ \leq & \int_{-\infty}^{\infty} E_0 |e_t(y,t)|(1+|y|)^{-3/2}\, dy \\ & + \int_0^t \int_{-\infty}^{+\infty} C\zeta(t)^2 |e_t(y,t-s)(-q,1)^{\mathrm{tr}} e^{-\eta y^+}|\Psi(y,s)\, dy\, ds \\ & + \int_0^t \int_{-\infty}^{+\infty} C\zeta(t)^2 |e_{yt}(y,t-s)|\Psi(y,s)\, dy\, ds \\ \leq & \, C(E_0 + \zeta(t)^2)(1+t)^{-1}. \end{aligned}$$

Dividing by $(\theta + \psi_1 + \psi_2)(x,t)$ and $(1+t)^{-1}$, respectively, we obtain (8.19) as claimed.

From (8.19), we obtain global existence, with $\zeta(t) \le 2CE_0$. From the latter bound and the definition of ζ in (8.18) we obtain the first two bounds of (8.5). It remains to establish the third bound, expressing convergence of phase δ to a limiting value $\delta(+\infty)$.

By Lemmas 8.3–8.4 together with the previously obtained bounds (8.22) and $\zeta \le CE_0$, and the definition (8.18) of ζ, the formal limit

$$\begin{aligned}
\delta(+\infty) := &\int_{-\infty}^{\infty} e(y,+\infty)U_0(y)\,dy \\
&+\int_0^t\int_{-\infty}^{\infty} |e(y,t-s)(-q,1)^{\mathrm{tr}}||r(U)(y,s)|\,dy\,ds \\
&+\int_0^{+\infty}\int_{-\infty}^{+\infty} e_y(y,+\infty)(Q(U)+\dot\delta U)(y,s)\,dy\,ds \\
\le &\int_{-\infty}^{\infty} |e(y,+\infty)|E_0(1+|y|)^{-3/2}\,dy \\
&+\int_0^t\int_{-\infty}^{+\infty} |e(y,t-s)(-q,1)^{\mathrm{tr}}e^{-\eta y^+}|CE_0\Psi(y,s)\,dy\,ds \\
&+\int_0^{+\infty}\int_{-\infty}^{+\infty} |e_y(y,+\infty)|CE_0\Psi(y,s)\,dy\,ds \\
\le &\, CE_0
\end{aligned}$$

is well-defined, as the sum of absolutely convergent integrals.

Applying Lemmas 8.3–8.4 a final time, we obtain

$$\begin{aligned}
|\delta(t)-\delta(+\infty)| \le &\int_{-\infty}^{\infty} |e(y,t)-e(y,+\infty)||U_0(y)|\,dy \\
&+\int_0^t\int_{-\infty}^{\infty} |\Big(e(y,t-s)-e(y,+\infty)\Big)(-q,1)^{\mathrm{tr}}||r(U)(y,s)|\,dy\,ds \\
&+\int_0^t\int_{-\infty}^{+\infty} |e_y(y,t-s)-e_y(y,+\infty)||(Q(U)+\dot\delta U)(y,s)|\,dy\,ds \\
&+\int_t^{+\infty}\int_{-\infty}^{+\infty} |e_y(y,+\infty)||(Q(U)+\dot\delta U)(y,s)|\,dy\,ds \\
\le &\int_{-\infty}^{\infty} |e(y,t)-e(y,+\infty)|E_0(1+|y|)^{-3/2}\,dy \\
&+\int_0^t\int_{-\infty}^{\infty} |\Big(e(y,t-s)-e(y,+\infty)\Big)(-q,1)^{\mathrm{tr}}e^{-\eta y^+}|CE_0\Psi(y,s)\,dy\,ds \\
&+\int_0^t\int_{-\infty}^{+\infty} |e_y(y,t-s)-e_y(y,+\infty)|CE_0\Psi(y,s)\,dy\,ds \\
&+\int_t^{+\infty}\int_{-\infty}^{+\infty} |e_y(y,+\infty)|CE_0\Psi(y,s)\,dy\,ds \\
\le &\, CE_0(1+t)^{-1/2},
\end{aligned}$$

establishing the remaining bound and completing the proof. $\square$

Proof of Theorem 1.5. Immediate from Proposition 8.1, by integration of bounds (8.5). □

Remark 8.7. Proposition 8.1 gives a time-asymptotic description of perturbation U as a superposition of algebraically decaying signals propagating along outgoing undamped characteristic directions. A brief examination reveals that these consist entirely of *fluid dynamical modes*, since reactive modes propagate always inward from the positive x side, and as damped outgoing modes on the negative x side. Recall that fluid modes lie asymptotically along direction $R_f^{\pm} = (*, 0)^{\rm tr}$ with vanishing z-component. Taking account of this fact, together with the faster decay rate of $(0,1)\tilde{G}(1,0)^{\rm tr}$ stated in Proposition 7.1, we could by essentially the same argument used to prove Proposition 8.1 establish the refined result that *the z-component of perturbation U decays faster than the u-component*, reflecting the physical picture that the fluid is in each case (weak or strong detonation or deflagration) swept through the traveling wave, burning completely in the high-temperature region in its wake. However, we do not determine precise bounds here.

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